Alejandro Badel

Homework 1, Macro II Jonathan Heathcote : Answer

1) a) The set of Pareto Efficient allocations coincides with the solutions to the planning problem. By the Inada conditions, solutions to this problem are always interior. Also, by strict concavity of the planners objective, the FOC are necessary and sufficient.

$$\begin{aligned} \frac{u'\left(c_t^1\right)}{u'\left(c_t^2\right)} &= \frac{1-\alpha}{\alpha} \\ c_t^1 + c_t^2 &= e_t^1 + e_t^2 := e_t \end{aligned}$$

Using the functional form for $u(.)$, we get:

$$\left(\frac{c_t^1}{c_t^2}\right)^{-\gamma} = \frac{1-\alpha}{\alpha}$$

Imposing the resource constraint:

$$\begin{aligned} \text{Pareto} &= \{\left(c_t^{p2}, c_t^{p1}\right) \in R_+^2 \text{ such that} \\ c_t^{p1} &= \frac{\alpha^{\frac{1}{\gamma}}}{\alpha^{\frac{1}{\gamma}} + (1-\alpha)^{\frac{1}{\gamma}}} e_t \\ c_t^{p2} &= \frac{(1-\alpha)^{\frac{1}{\gamma}}}{\alpha^{\frac{1}{\gamma}} + (1-\alpha)^{\frac{1}{\gamma}}} e_t \\ \alpha &\subset [0,1]\} \end{aligned}$$

Note that consumption varies across periods but the ratio of consumption remains constant.

b) Denoting the Lagrange multiplier for the resource constraint of the Planner's problem above by $\mu_t/2$, and recalling one of the FOC,

$$\beta^t \left(c_t^1\right)^{-\gamma} = \frac{\mu_t}{2\alpha}$$

The FOC of the decentralized sequential markets problems are

$$\left(c_t^1\right)^{-\gamma} = \lambda_t^1 p_t$$

thus, we can let

$$\begin{aligned} p_t^{eq} &= \mu_t \\ \lambda_t^1 &= \frac{1}{2\alpha} \end{aligned}$$

At the Pareto efficient allocation implied by a given α, we get

$$\beta^t \left(e_t \frac{(1-\alpha)^{\frac{1}{\gamma}}}{\alpha^{\frac{1}{\gamma}} + (1-\alpha)^{\frac{1}{\gamma}}} \right)^{-\gamma} = \frac{p_t(\alpha)}{2\alpha}$$

$$p_t^{eq} = \frac{2\alpha}{(1-\alpha)} \frac{1}{\left(\alpha^{\frac{1}{\gamma}} + (1-\alpha)^{\frac{1}{\gamma}}\right)^{-\gamma}} e_t^{-\gamma}\beta^t$$

Note that letting $p_t^{eq} = \mu_t$ is equivalent to setting $p_t^{eq} = e_t^{-\gamma}\beta^t$ because equilibrium prices are homogeneous of degree one. (why?).

And impose the budget constraints by calculating the transfer functions at $t(\alpha^*) = 0$.

$$\begin{aligned} 0 &= \sum_{t=0}^{\infty} p_t^{eq} \left[c_t^{p1} - e_t^1 \right] \\ 0 &= \sum_{t=0}^{\infty} e_t^{-\gamma}\beta^t \left[c_t^{p1} - e_t^1 \right] \end{aligned}$$

Using the process for returns and the pareto efficient allocation

$$\begin{aligned} 0 &= \sum_{t=0}^{\infty} \beta^t e_t^{-\gamma} c_t^{p1} - \frac{e_{even}^{-\gamma} e_{even}^1}{1-\beta^2} - \frac{\beta e_{odd}^{-\gamma} e_{odd}^1}{1-\beta^2} \\ 0 &= \sum_{t=0}^{\infty} \beta^t \frac{e_t^{1-\gamma}\alpha^{\frac{1}{\gamma}}}{\alpha^{\frac{1}{\gamma}} + (1-\alpha)^{\frac{1}{\gamma}}} - \frac{e_{even}^{-\gamma} e_{even}^1 + \beta e_{odd}^{-\gamma} e_{odd}^1}{1-\beta^2} \end{aligned}$$

$$\begin{aligned} \frac{e_{even}^{1-\gamma} + \beta e_{odd}^{1-\gamma}}{1-\beta^2} \frac{\alpha^{\frac{1}{\gamma}}}{\alpha^{\frac{1}{\gamma}} + (1-\alpha)^{\frac{1}{\gamma}}} &= \frac{e_{even}^{-\gamma} e_{even}^1 + \beta e_{odd}^{-\gamma} e_{odd}^1}{1-\beta^2} \\ \frac{\alpha^{\frac{1}{\gamma}} + (1-\alpha)^{\frac{1}{\gamma}}}{\alpha^{\frac{1}{\gamma}}} &= \frac{e_{even}^{1-\gamma} + \beta e_{odd}^{1-\gamma}}{e_{even}^{-\gamma} e_{even}^1 + \beta e_{odd}^{-\gamma} e_{odd}^1} \end{aligned}$$

$$\begin{aligned} \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{\gamma}} &= \frac{e_{even}^{1-\gamma} + \beta e_{odd}^{1-\gamma}}{e_{even}^{-\gamma} e_{even}^1 + \beta e_{odd}^{-\gamma} e_{odd}^1} - 1 \\ \frac{1}{\alpha} &= \left(\frac{e_{even}^{1-\gamma} + \beta e_{odd}^{1-\gamma}}{e_{even}^{-\gamma} e_{even}^1 + \beta e_{odd}^{-\gamma} e_{odd}^1} - 1 \right)^{\gamma} + 1 \\ \alpha^* &= \left\{ \left(\frac{e_{even}^{1-\gamma} + \beta e_{odd}^{1-\gamma}}{e_{even}^{-\gamma} e_{even}^1 + \beta e_{odd}^{-\gamma} e_{odd}^1} - 1 \right)^{\gamma} + 1 \right\}^{-1} \end{aligned}$$

Substituting the values for endowments and parameters

$$
\begin{aligned}
\alpha^* &= \left\{ \left(\frac{2^{1-2} + (0.9)\,3^{1-2}}{2^{-2}\,(1.5) + (0.9)\,3^{-2}} - 1 \right)^2 + 1 \right\}^{-1} \\
\alpha^* &= 0.68113 \\
\text{share of 1} &: \frac{0.68113^{\frac{1}{2}}}{0.68113^{\frac{1}{2}} + (1 - 0.68113)^{\frac{1}{2}}} \\
\text{share of 2} &: \frac{(1 - 0.68113)^{\frac{1}{2}}}{0.68113^{\frac{1}{2}} + (1 - 0.68113)^{\frac{1}{2}}}
\end{aligned}
$$

Thus, the competitive allocation is

$$
\left\{ p_t^c : p_t^c = e_t^{-2} \beta^t \right\}_{t=0}^{\infty}
$$

$$
\begin{aligned}
\{ \left(c_t^{1c}, c_t^{2c} \right) &: \\
c_t^{1c} &= 0.59 e_t \\
c_t^{2c} &= 0.41 e_t \}_{t=0}^{\infty}
\end{aligned}
$$

c) Since the equilibrium prices and allocations satisfy

i) optimality condition of households (FOC, BC)

ii) budget constraints

we know by (i) and (ii) that they asatisfy optimality. And by (ii) we know they are feasible.

d) Agent 1 recieves a higher share of consumption in he competitive allocation. This reflects two things about agent 1's endowment stream: i) agent has a larger "present value" endowment, because he gets the large payment in period 0. ii) agent 1's endowment flow is "flatter", thus, he has a smaller need for trading compared to agent 1.

2) Making use on the equivalence of the Arrow-Debreu Eqbm. and the Sequential Markets Equilibrium, I use the results obtained for this equilibrium in the notes.

Pick any α, let prices be the lagrange multipliers of the planner's problem. Let $\lambda_1 = 1/2\alpha$, $\lambda_2 = 1/2\,(1-\alpha)$. Then, the FOC of both agents are satisfied by the pareto efficient consumption allocation. now, the budget constraint of agent 1 as a function of initial assets is:

$$
\begin{aligned}
t_1\,(a_0) &= \sum_{t=0}^{\infty} \beta^t \left[c_t^{p1} - e_t^1 \right] - a_0 \\
t_2\,(a_0) &= \sum_{t=0}^{\infty} \beta^t \left[c_t^{p2} - e_t^2 \right] + a_0
\end{aligned}
$$

Replacing the Pareto allocation and the values for endowments:

$$
\begin{aligned}
t_1(a_0) &= \frac{2\alpha}{1-\beta} - \frac{2}{1-\beta^2} - a_0 \\
t_2(a_0) &= \frac{2(1-\alpha)}{1-\beta} - \frac{2\beta}{1-\beta^2} + a_0
\end{aligned}
$$

Set $t_1(a_0^*) = 0$

$$
a_0^* = \frac{2\alpha}{1-\beta} - \frac{2}{1-\beta^2}
$$

inserting this into $t_2(a_0)$ we get

$$
\begin{aligned}
t_2(a_0^*) &= \frac{2(1-\alpha)}{1-\beta} - \frac{2\beta}{1-\beta^2} + \frac{2\alpha}{1-\beta} - \frac{2}{1-\beta^2} \\
t_2(a_0^*) &= \frac{2}{(1-\beta)} - \frac{2\beta}{1-\beta^2} - \frac{2}{1-\beta^2} \\
&= \frac{2\left(1-\beta^2\right) - 2\beta(1-\beta) - 2(1-\beta)}{(1-\beta)\left(1-\beta^2\right)} \\
&= \frac{2-2\beta^2-2\beta+2\beta^2-2\beta}{(1-\beta)\left(1-\beta^2\right)} = 0
\end{aligned}
$$

Therefore budget constraints are satisfied. Feasibility is satisfied by Pareto efficiency.