Graduate Macro II, Homework 2

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Due in class on Thursday February 16th 2006

Consider the neoclassical growth model in discrete time. In the two examples below, you are asked to solve for efficient allocations by considering social planner's problems.

1. Assume the planner's preferences are given by $\sum_{t=0}^{T} \beta^t u(c_t)$ where $u(c_t) = \ln(c_t)$, and the technology is

$$\begin{aligned} c_t + i_t &= y_t \qquad \forall t \geq 0 \\ y_t &= k_t^\theta \qquad \forall t \geq 0 \\ k_{t+1} &= (1-\delta)k_t + i_t \qquad \forall t \geq 0 \\ c_t &\geq 0,\ k_t \geq 0 \qquad \forall t \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

   (a) Describe a set of equations that implicitly defines a solution to this planning problem.

   (b) Assume the following parameter values: $\beta = 0.96$, $T = 100$, $\theta = 0.36$, $\delta = 0.08$. Assume $k_0 = 1$. Solve for the efficient allocation $\{c_t^*, k_t^*\}_{t=0}^{T}$. I suggest you use the following "shooting method":

      i. Guess bounds such that $c_l^1 \leq c_0^* \leq c_h^1$, and make an initial guess for c_0^*, denoted c_0^1, where $c_0^1 = 0.5\left(c_l^1 + c_h^1\right)$

      ii. Use the set of equations defining a solution to the problem to compute k_{T+1} given c_0^1 and k_0

      iii. Use "bisection" to update the guess for c_0. In particular, if $k_{T+1} > 0$ then set $c_l^2 = c_0^1$, else if $k_t < 0$ for any $t \leq T+1$ then set $c_h^2 = c_0^1$. Then set $c_0^2 = 0.5\left(c_l^2 + c_h^2\right)$.

      iv. Go to step (ii), replacing c_0^1 with c_0^2.

      v. Repeat until $|k_{T+1}| < 0.01$.

   (c) Plot two graphs showing $\{c_t^*\}_{t=0}^{T}$ and $\{k_t^*\}_{t=0}^{T+1}$. Comment on the time series correlation between the level of the capital stock, and the growth rate of consumption.

2. Now consider the infinite horizon version of the economy described above, *ie* set $T = \infty$.

   (a) Write down the recursive formulation of the planner's problem in this case.

   (b) Solve for the steady state values for capital, output and consumption.

   (c) Solve (approximately) for the value function that solves the recursive problem, $v(k)$, and the optimal policy function $k' = g(k)$. I suggest you use the following "discrete state space" approach:

      i. Construct a equi-spaced grid on capital, $k \in K = \{k_1, k_2, ..., k_N\}$. I suggest using $k_1 = 0.01$, $k_N = 10$, $N = 100$.

      ii. Make an initial guess for a value function defined over these points, $v^1(k)$. For example set $v^1(k_i) = 0 \ \forall i = 1, ..., N$.

      iii. For each point k_i in the grid on capital, beginning with k_1, solve the following maximization problem: Find the value for $k' \in K$ that is feasible and that maximizes the right hand side of the Bellman equation at $k = k_i$ when the continuation value is given by $v^1(k')$. Save for step (iv) the decision rule, $g^2(k_i)$, and the implied new value for the value function, $v^2(k_i)$.

      iv. Update the right hand side of the Bellman equation by setting continuation values equal to $v^2(k_i) \ \forall i$.

      v. Return to step (iii). Repeat until $\sum_{i=1}^{N} \left| v^n(k_i) - v^{n-1}(k_i) \right| \leq 0.00001$.

   (d) Plot k against $g(k)$ and plot $\ln(k)$ against $\ln(g(k))$

   (e) Is the optimal decision rule $g(k)$ closer to linear (*ie* $k' = \alpha + \beta k$) or log-linear (*ie* $\ln(k') = \alpha + \beta \ln(k)$)?

   (f) Suppose we start at $k_0 = 1$ (or the closest point on your grid). Plot the sequence $\{k_t\}_{t=0}^{T}$ implied by $g(k)$.

      i. Do we reach the steady state you solved for in (b)?

      ii. How does this sequence compare to your answer for Question 1?