# Homework 3 - Partial Answers

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Due in Class on Tuesday February 28th

In class we outlined two versions of the stochastic growth model: a planner's problem, and an Arrow-Debreu competitive equilibrium. We were working towards showing that allocations in the two setups would be identical.

1. (a) Complete the proof that the sets of equations that characterize (i) the solution to the planner's problem, and (ii) the competitive equilibrium are identical, and thus that one can solve for equilibrium allocations by solving the planner's problem.

   This is pretty straightforward following the class notes, and the textbook.

   (b) Now consider the following twist on the economy we described in class. Income (from both labor and capital) is taxed at rate $\tau_t$, where $0 < \tau_t < 1$. There is no allowance for depreciation: thus the typical consumer's budget constraint (in the sequence of markets formulation, without state-contingent claims) is

$$c_t + k_{t+1} = (1 - \tau_t)(r_t k_t + w_t n_t) + (1 - \delta)k_t$$

   Revenues are used for non-valued government purchases $G$. Consider (i) a planner's problem in which the planner has to set aside a constant amount $G$ of output each period for government purchases, and (ii) a competitive equilibrium in which the tax rate $\tau_t$ is such that at each date equilibrium revenue is equal to the same amount $G$.

      i. Describe the planner's problem and the competitive equilibrium, and the two sets of equations characterizing (i) the planner's solution and (ii) the equilibrium.

      The planner's inter-temporal first order condition reduces to something like

$$u_c(c_t, l_t) = \beta E_t \left[ u_c(c_{t+1}, l_{t+1}) \left( 1 - \delta + MPK_{t+1} \right) \right]$$

      The corresponding condition from the consumer's problem in the competitive equilibrium is

$$u_c(c_t, l_t) = \beta E_t \left[ u_c(c_{t+1}, l_{t+1}) \left( 1 - \delta + r_{t+1}(1 - \tau_{t+1}) \right) \right]$$

From the firm's problem,

$$r_{t+1} = MPK_{t+1}$$

Thus the two inter-temporal FOCs are different.
The planner's intra-temporal FOC is

$$u_c(c_t, l_t)MPN_t = u_l(c_t, l_t)$$

The consumer's first order condition is

$$u_c(c_t, l_t)(1 - \tau_t)w_t = u_l(c_t, l_t)$$

From the firm's problem,

$$w_t = MPN_t$$

Thus the two inter-temporal FOCs are different

ii. In general, are allocations the same in each case?

No - the set of equations characterizing the planner's solution and the competitive equilibrium are different. A set of values for $c_t$, $l_t$ and $k_t$ that satsify the planner's first order conditions will not satisfy the competitive conditions.

iii. Now suppose the utility function takes the form

$$u(c, n) = \ln(c) + v(1 - n)$$

where $v(.)$ is strictly increasing and strictly concave.
Are allocations the same in the competitive equilibrium and the planner's problem in this case? If so, why? If not, what additional policy instruments would the government need in the decentralized economy to achieve the allocation that solves the planner's problem?

The inter-temporal FOCs remain different. One might suspect that with balanced growth preferences, the labor distortion would not matter for allocations, but in fact the intra-temporal first order conditions will also be different. Consider the competitive equilibrium condition (asumming constant returns to scale production):

$$\begin{aligned}
u_c(c_t, l_t)(1 - \tau_t)w_t &= u_l(c_t, l_t) \\
\frac{(1 - \tau_t)w_t}{c_t} &= v'(1 - n_t) \\
\frac{(1 - \tau_t)w_t}{(1 - \tau_t)(r_t k_t + w_t n_t) - x_t} &= v'(1 - n_t) \\
\frac{(1 - \tau_t)MPN_t}{(1 - \tau_t)y_t - x_t} &= v'(1 - n_t) \\
\frac{(1 - \tau_t)MPN_t}{y_t - x_t - g} &= v'(1 - n_t)
\end{aligned}$$

The corresponding condition for the planner's problem simplfies to

$$\begin{aligned} u_c(c_t, l_t) MPN_t &= u_l(c_t, l_t) \\ \frac{MPN_t}{c_t} &= v'(1 - n_t) \\ \frac{MPN_t}{y_t - x_t - g} &= v'(1 - n_t) \end{aligned}$$

To achieve the allocation that solves the planner's problem, the government would need access to lump-sum taxes

(c) Consider the competitive equilibrium described in part (*b*). Suppose the period utility function for the representative consumer is

$$u(c, l) = \ln(c) + \psi \ln(1 - n)$$

Suppose output, produced by a representative firm, is given by

$$\begin{aligned} y_t &= z_t F(k_t, n_t) \\ F(k, n) &= k^\theta n^{1-\theta} \end{aligned}$$

Suppose individuals discount at rate $\beta$, and capital depreciates at rate $\delta$. Consider the non-stochastic steady state for this economy. Assume that in the non-stochastic steady state $z = 1$. Suppose we want to calibrate the economy to replicate the following facts:

i. Two-thirds of income goes to labor:

$$\frac{wn}{wn + rk} = \frac{2}{3}$$

ii. People work one-third of the time endowment

$$n = \frac{1}{3}$$

iii. Government spending is 20% of output

$$\frac{G}{Y} = 0.2$$

iv. Investment is 15% of output

$$\frac{x}{Y} = 0.15$$

v. The annual after-tax return to capital, net of depreciation, is 4%

$$(1 - \tau) r - \delta = 0.04$$

What values for $\beta$, $\psi$, $\theta$, $\delta$ and $\tau$ does this calibration imply?
This is quite straightforward

$$\frac{wn}{wn+rk} = 1-\theta = \frac{2}{3} \Rightarrow \theta = \frac{1}{3}$$

$$\frac{G}{Y} = 0.2 \Rightarrow \tau = 0.2$$

Now take the intra-temporal first order condition:

$$\begin{aligned}
\frac{w(1-\tau)}{c} &= \psi\frac{1}{1-n} \\
\frac{(1-\theta)\frac{Y}{n}(1-\tau)}{(1-\tau)Y-X} &= \psi\frac{1}{1-n} \\
\frac{1}{n}\frac{(1-\theta)(1-\tau)}{(1-\tau)-\frac{X}{Y}} &= \psi\frac{1}{1-n} \\
\psi &= \frac{\left(1-\frac{1}{3}\right)}{\frac{1}{3}}\frac{(1-\frac{1}{3})(1-0.2)}{(1-0.2)-0.15} = 1.641
\end{aligned}$$

(where in the second line I used the relations $w = (1-\theta)\frac{Y}{N}$ and $c = (1-\tau)Y - X$)
Now take the inter-temporal first order condition in the non-stochastic steady state

$$\begin{aligned}
u'(c) &= \beta u'(c)(1+r(1-\tau)-\delta) \\
1 &= \beta(1+r(1-\tau)-\delta) \\
1 &= \beta(1+0.04) \\
\beta &= \frac{1}{1.04}
\end{aligned}$$

But using the same equation

$$1 = \beta(1+\frac{\theta Y}{K}(1-\tau)-\delta)$$

and

$$\frac{X}{Y} = \frac{\delta K}{Y} = 0.15$$

so

$$1 = \frac{1}{1.04}(1+\frac{\frac{1}{3}\delta}{0.15}(1-0.2)-\delta) \Rightarrow \delta = 0.05143$$

Thus the implied capital output ratio is

$$\frac{0.15}{\delta} = \frac{0.15}{0.05143} = 2.917$$

(d) Define a recursive competitive equilibrium for the economy with taxation. Is the state vector the same as the one for the non-distorted economy?

Let's suppose that the process for TFP is defined by the transition probability matrix for $z \in Z$ given by $\Pi$ where $\pi(z_j|z_i)$ is the probability that $z_{t+1} = z_j$ given that $z_t = z_i$.
The state vector is the same as for the non-distorted economy: $(z, K, k)$.
Define the household's problem recursively:

$$V(z, K, k) = \max_{c, n^s, k'} \left\{ u(c, 1 - n^s) + \beta \sum_{z' \in Z} \pi(z'|z) V(z', K', k') \right\}$$

subject to the constraints

$$\begin{aligned} c + k' &= (1 - \tau(z, K))(r(z, K)k + w(z, K)n^s) + (1 - \delta)k \\ c &\geq 0 \\ n^s &\geq 0 \\ n^s &\leq 1 \end{aligned}$$

and taking as given price functions $r(z, K)$ a.nd $w(z, K)$, a law of motion for capital $K' = G(z, K)$, and a tax function $\tau(z, K)$
Since the production function is constant returns to scale we will imagine a single representative price-taking firm
This firm solves a static maximization problem

$$W(z, K) = \max_{n^d, k^d} \left\{ zF(k^d, n^d) - w(z, K)n^d - r(z, K)k^d \right\}$$

subject to

$$k^d, n^d \geq 0$$

taking as given prices $w(z, K)$ and $r(z, K)$. (note there is no individual state for the firm - the firm has no assets at the start of the period)
A recursive competitive equilibrium is a set of pricing functions $w(z, K)$ and $r(z, K)$, decision rules $n^s(z, K, k)$, $c(z, K, k)$, $k'(z, K, k)$, $n^d(z, K)$, $k^d(z, K)$, a value function $v(z, K, k)$, a tax function $\tau(z, K)$, and a law of motion $G(z, K)$ such that:

2. Given $w$ $r$ $G$ and $\tau$, the functions $n^s$, $c$ and $k'$ solve the household problem, and $v$ is the associated value function

3. Given $w$ and $r$, the functions $k^d$ and $n^d$ solve the firm's problem

4. Markets clear:

   (a) $n^s(z, K, K) = n^d(z, K)$ (rental market for labor)

(b) $k^d(z, K) = K$ (rental market for capital)

(c) $c(z, K, K) + G + k'(z, K, K) = zF(K, n^d(z, K)) + (1 - \delta)K$ (goods market)

5. Perceived law of motion for aggregate capital consistent with individual decisions: $G(z, K) = k'(z, K, K)$ (rational expectations)

6. Government budget constraint is satisfied $\tau(z, K)zF(K, n^d(z, K)) = G$