Alejandro Badel.
Macro II, Jonathan Heatchote. Homework 5 Answers.

1)
Recursive formulation of the individual's maximization problem:
state: (a, w) where a represents beggining of period bond holdings and w is the current realization of the wage process.
control: (n, a') where n are hours worked and $a\prime$ are end of period bond holdings.
feasible set: $\Gamma(a, w) = \{(n, a') : wn + (1 + r) a - a' \geq 0, a' \geq 0\}$
Bellman Equation:

$$\begin{aligned} V(a, w) &= \max_{a' \in \Gamma(a,w)} \left\{ \frac{1}{1-\gamma} \left(wn + (1+r) a - a' - \frac{\psi n^{1+1/\varepsilon}}{1+1/\varepsilon} \right)^{1-\gamma} + \beta E \left[V(a', w') | w \right] \right\} \\ & \quad w' \text{ random draw from } \pi(w'|w) \end{aligned}$$

2)
First order conditions
Intertemporal

$$u_c \geq \beta (1 + r) E [u'_c | w]$$

Intratemporal

$$wu_c = -u_n$$

3)
From the Intratemporal FOC we have

$$\begin{aligned} w &= \psi n^{1/\varepsilon} \\ \ln n &= -\varepsilon \ln \psi + \varepsilon \ln w \end{aligned}$$

We can see that the elasticity of worked hours with respect to wages is given by ε.

4)
We could assume for simplicity that the wage distribution starts off at its stationary value, this would make the planner's problem invariant over time. Mantaining the assumption that the initial distribution is uniform (50 50), we can formulate the planner's problem recursively as follows:
State: p current fraction of agents with high wages (w_h).
Control: (c_h, n_h, n_l) consumption by the high wage agents, hours worked by the high and low wage agents.
Feasible set: $\Gamma(p) = \{(c_h, n_h, n_l) \gg 0 : c_h \leq p w_h n_h + (1 - p) w_l n_l\}$

Transition equation: $\begin{bmatrix} p' \\ 1-p' \end{bmatrix} = G(p) = [p, 1-p]\,\pi$
Bellman Equation:

$$\begin{aligned} V(p) &= \max_{c_h \in \Gamma(p)} \{pu(c_h, n_h) + (1-p)\,u(c_l, n_l) + \beta V(p')\} \\ (1-p)\,c_l &: = pw_h n_h + (1-p)\,w_l n_l - pc_h \\ p' &= G(p)_1 \end{aligned}$$

5)
First order conditions of the Planner's Problem:
a)
$[c_h]$:

$$u_1(c_h, n_h) = u_1(c_l, n_l)$$

$[n_h]$:

$$u_2(c_h, n_h) = -w_h u_1(c_l, n_l)$$

$[n_l]$:

$$u_2(c_l, n_l) + w_l u_1(c_l, n_l) = 0$$

Substituting $[n_l]$ into $[n_h]$

$$u_2(c_h, n_h) = \frac{w_h}{w_l} u_2(c_l, n_l)$$

Dividing the equation above by the $[c_h]$ condition

$$\begin{aligned} n_h^{1/\varepsilon} &= \frac{w_h}{w_l} n_l^{1/\varepsilon} \\ n_h &= \left(\frac{w_h}{w_l}\right)^{\varepsilon} n_l \end{aligned}$$

Since $w_h/w_l > 1$, $\varepsilon > 0$ implies that high wage agents work more hours than low wage workers.

By $[c_h]$, this implies that $c_h > c_l$. The planner does not equalize consumption at a given date.

b) Suppose that there is no transfer between agents, then $c_l = w_l n_l$ and $c_h = w_h n_h$.

by the expression for n_h above we have

$$c_h = w_h \left(\frac{w_h}{w_l}\right)^{\varepsilon} n_l$$

Under this assumption, we have

$$\begin{aligned} u_1(c_h, n_h) &= \left(w_h \left(\frac{w_h}{w_l} \right)^{\varepsilon} n_l - \psi \left(\frac{w_h}{w_l} \right)^{\varepsilon+1} \frac{n_l^{1+1/\varepsilon}}{1+1/\varepsilon} \right)^{-\gamma} \\ &= \left(\frac{w_h^{\varepsilon+1}}{w_l^{\varepsilon+1}} \right)^{-\gamma} \left(w_l n_l - \psi \frac{n_l^{1+1/\varepsilon}}{1+1/\varepsilon} \right)^{-\gamma} \\ &< \left(w_l n_l - \psi \frac{n_l^{1+1/\varepsilon}}{1+1/\varepsilon} \right)^{-\gamma} = u_1(c_l, n_l) \end{aligned}$$

This contradicts the first order condition $[c_h]$.

Since $u_1(c_l, n_l)$ is decreasing in c_l optimality (i.e equality in the expresion above) requires a positive transfer of resources from the high wage agents to the low wage agents.

6) Assume $a_0 = 0$. Now, at $a = 0$, and if r is sufficiently low such that $a' = 0$ the following first order condition must condition must hold:

$$(wn)^{-\gamma} \leq (1+r)\beta E\left[(w'n')^{-\gamma} | w \right]$$

The intra temporal condition is

$$\begin{aligned} wc^{-\gamma} &= \psi n^{1/\varepsilon} \\ w(wn)^{-\gamma} &= \psi n^{1/\varepsilon} \\ n &= \left(\frac{w^{1-\gamma}}{\psi} \right)^{1/(\gamma+1/\varepsilon)} \end{aligned}$$

So we can pin down n and $n\prime$ and substitute them in the intertemporal condition

$$\begin{aligned} \left(w \left(\frac{w^{1-\gamma}}{\psi} \right)^{1/(\gamma+1/\varepsilon)} \right)^{-\gamma} &\geq (1+r)\beta E\left[\left(w \left(\frac{w'^{1-\gamma}}{\psi} \right)^{1/(\gamma+1/\varepsilon)} \right)^{-\gamma} | w \right] \\ \frac{\psi^{\gamma/(\gamma+1/\varepsilon)}}{w^{-(\varepsilon+1)/(\varepsilon+1/\gamma)}} &\geq (1+r)\beta E\left[\frac{\psi^{\gamma/(\gamma+1/\varepsilon)}}{w'^{-(\varepsilon+1)/(\varepsilon+1/\gamma)}} | w \right] \end{aligned}$$

This inequality must hold for all possible values of w.

Since the argument inside the expectation is decreasing in w', a sufficient condition for both inequalities to hold can be obtained by setting $w = w_h$ and $w\prime = w_l$ in the previous inequality; this will make the LHS as small as it can be, and the RHS as large as it can be. (Note that a highest upper bound can be achived by using conditions for w_l and w_h explicitly. I took a shortcut here).

This substitution yields

$$\left(\frac{w_l}{w_h}\right)^{(\varepsilon+1)/(\varepsilon+1/\gamma)} \geq \beta\left(1+r\right)$$

$$r \leq \frac{1}{\beta}\left(\frac{w_l}{w_h}\right)^{(\varepsilon+1)/(\varepsilon+1/\gamma)} - 1$$

Thus, the lower bound for r is given by

$$\overline{r}=\left(\frac{w_l}{w_h}\right)^{(\varepsilon+1)/(\varepsilon+1/\gamma)} -1$$

7) When $r \leq \overline{r}$, we have that bond holdings are zero and $c = wn$, then, the intra temporal condition is

$$n = \left(\frac{w^{1-\gamma}}{\psi}\right)^{1/(\gamma+1/\varepsilon)}$$

Therefore, n is increasing in w if $\gamma < 1$ (i.e when there is not too much curvature in the ultility function and the preference for consumption smoothing is not too strong).

8) Assuming $\gamma > 1$ and $r \leq \overline{r}$ we get that hours and wages are negatively correlated and that $a = a' = 0$.

If hours and consumption are negatively correlated, the agent is smoothing consumption by working more in low wage periods.

Quoting prof. Heathcote: "With higher interest rates, the agent will use savings to smooth consumption, and adjust hours to make hay when the sun shines - increasing hours when wages go up - thereby increasing the wage hour correlation"