Simple Consumption / Savings Problems (based on Ljungqvist & Sargent, Ch 16, 17)

Jonathan Heathcote

updated, March 2006

1. The household's problem

$$\max E \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

subject to for all t

$$c_t + a_{t+1} = we_t + (1+r)a_t$$
$$a_{t+1} \geq -\phi$$
$$c_t \geq 0$$

given a_0, where $e_t \in E = \{e_1, e_2, e_3, ..., e_N\}$ follows a first order Markov process with transition probabilities given by an $N \times N$ transition probability matrix Π. The utility function u is strictly increasing, strictly concave, twice continuously differentiable.

2. The value function and first order condition

The state variables for the household are a and e. In recursive form, the household's problem is described by the following Bellman's equation.

$$V(a,e) = \max_{a'} \left[u(c) + \beta \sum_{e'} \pi(e'|e) V(a', e') \right]$$

such that

$$c = we + (1+r)a - a'$$
$$a' \geq -\phi$$

FOC wrt a'

$$-u'(c) + \beta \sum_{e'} \pi(e'|e) V_1(a', e') \leq 0$$

Envelope condition

$$V_1(a,e) = (1+r)u'(c)$$

Euler equation

$$u'(c) \geq \beta(1+r) \sum_{e'} \pi(e'|e) u'(c') \qquad = \text{if } a' \geq -\phi$$

3. THE BORROWING CONSTRAINT

Where does ϕ come from?

Impose $c_t \geq 0$ and iterate the budget constraint forwards

$$c_t \geq 0 \Rightarrow we_t + (1+r)a_t - a_{t+1} \geq 0$$

$$\Rightarrow a_t \geq \frac{a_{t+1} - we_t}{(1+r)}$$

$$a_t \geq \frac{\frac{a_{t+2} - we_{t+1}}{(1+r)} - we_t}{(1+r)}$$

$$a_t \geq -\frac{1}{(1+r)} \sum_{j=0}^{\infty} we_{t+j} (1+r)^{-j} = -\sum_{j=1}^{\infty} we_{t+j-1} (1+r)^{-j}$$

The constraint is more naturally expressed as a limit on a_{t+1}, so updating one period gives

$$a_{t+1} \geq -\sum_{j=1}^{\infty} we_{t+j} (1+r)^{-j}$$

Now the right hand side of this equation is a random variable. We could introduce an expectation sign, but then if a household borrows up to the limit and receives a sequence of bad shocks then it will be unable to repay and maintain positive consumption. Thus it seems reasonable to impose that the constraint holds almost surely in which case we can replace e_{t+j} with e_1 which gives

$$a_{t+1} \geq -\frac{we_1}{r}$$

Of course we could easily impose an additional exogenous constraint that says that debt cannot exceed some limit b. In this case the effective borrowing limit will be given by

$$\phi = \min\left(b, \frac{we_1}{r}\right)$$

4. NON-STOCHASTIC CASE

4.1. $\beta(1+r) = 1$ and $\phi = 0$. Euler equation simplifies to

$$u'(c) \geq u'(c')$$

which, since u is strictly concave, implies

$$c' \geq c$$

Along the optimal path either $c_t^* = c_{t-1}^*$ or $c_t^* > c_{t-1}^*$ and $c_{t-1}^* = we_t$.

- Why would the household never choose $c_t^* < c_{t-1}^*$? (we discussed this in class)

The optimal consumption path converges to a positive finite limit when the agent reaches the period with the highest annuity value of the remainder of the income process.

$$\overline{c}\frac{1+r}{r} = \sup_t \left(\sum_{j=0}^{\infty} (1+r)^{-j} w e_{t+j} \right)$$

The LHS of this equation is the present value of consumption given constant consumption $\overline{c}$, the RHS is the present value of lifetime earnings. The intuition for this result is as follows. As time evolves every so often the borrowing constraint binds. Each time it binds the agent has zero assets at the start of the next period ($a_{t+1} = \phi = 0$). If the agent faced an arbitrarily loose borrowing constraint, she would like to set consumption at this date $(t+1)$ equal to a constant, the present value of lifetime earnings. At the $t+1$ when lifetime earnings are maximized, such a strategy is feasible, even when $\phi = 0$ (proof in LS).

4.2. ϕ is natural borrowing constraint in non-stochastic case. In the non-stochastic case, if the borrowing constraint is the natural borrowing constraint, it will never be binding. The natural borrowing constraint in this case is

$$a_{t+1} \geq -\sum_{j=1}^{\infty} w e_{t+j} (1+r)^{-j}$$

If the constraint never binds, consumption will be constant from any initial date t onwards:

$$c_t \frac{1+r}{r} = \overline{c}\frac{1+r}{r} = \sum_{j=0}^{\infty} (1+r)^{-j} w e_{t+j} + (1+r) a_t$$

The LHS is the present value of lifetime consumption given $c_t = \overline{c}$ for all t. The RHS is the present value of lifetime earnings plus initial wealth at date t. Given this strategy, at each date t, it is straightforward to show that if a_t satisfies the natural borrowing constraint, so does a_{t+1} :

$$\begin{aligned} a_{t+1} &= we_t + (1+r)a_t - c_t \\ &= we_t + (1+r)a_t - \frac{r}{1+r}\left(\sum_{j=0}^{\infty}(1+r)^{-j}we_{t+j} + (1+r)a_t\right) \\ &= we_t + a_t - r\left(\sum_{j=1}^{\infty} we_{t+j-1}(1+r)^{-j}\right) \\ &\geq we_t - \sum_{j=1}^{\infty} we_{t+j-1}(1+r)^{-j} - r\left(\sum_{j=1}^{\infty} we_{t+j-1}(1+r)^{-j}\right) \\ &= we_t - (1+r)\left(\frac{we_t}{1+r} + \sum_{j=2}^{\infty} we_{t+j-1}(1+r)^{-j}\right) \\ &= -\sum_{j=1}^{\infty} we_{t+j}(1+r)^{-j} \end{aligned}$$

(in the second line, I plugged in the expression for c_t, in the fourth I used the fact that a_t has to satisfy the natural borrowing constraint)

5. Stochastic case

- Can we expect similar results in the stochastic case?

No. If $\beta(1+r) = 1$ and $\phi = 0$ then household asset holdings and consumption converge almost surely to infinity. We will show this three different ways, under different sets of assumptions

5.1. Approach 1: assume marginal utility is convex.

Assume that $u'''(c) > 0$ (ie assume the function $u'(c)$ is convex)

The household's first order condition is

$$u'(c) \geq \beta(1+r)\sum_{e'}\pi(e'|e)u'(c'(e')) = \sum_{e'}\pi(e'|e)u'(c'(e'))$$

We want to show that consumption is increasing in expected terms. If we can do so for the case in which the FOC is an equality, it will follow immediately that consumption must be increasing in expected terms when the constraint is binding (since current consumption when the constraint is binding is lower than when it is not binding).

Now the marginal utility of consumption is a convex function. Thus by Jensen's inequality

$$\sum_{e'}\pi(e'|e)u'(c'(e')) \geq u'\left(\sum_{e'}\pi(e'|e)c'(e')\right)$$

Combining these two inequalities

$$u'(c) \geq u'\left(\sum_{e'} \pi(e'|e)c'(e')\right)$$

But the marginal utility of consumption is decreasing (by concavity) so

$$c \leq \sum_{e'} \pi(e'|e)c'(e')$$

In other words consumption tomorrow is larger than consumption today in expected terms. Since this applies at every date, consumption will continually ratchet upwards. A corollary is that consumption will never converge to a constant level, contrary to the case without uncertainty.

5.2. Approach 2: Martingales. A stochastic process $\{q_t\}$ that has the property that

$$q_t = E_t\left(q_{t+1}\right)$$

is called a martingale. In words, the best predictor of the variables future value is its current value.

A stochastic process $\{q_t\}$ that has the property that

$$q_t \geq E_t\left(q_{t+1}\right)$$

is called a supermartingale.

There is a theorem (Doob 1953) that say that non-negative supermartingales converge almost surely to a non-negative value.

A simple example of a martingale: the gambler's ruin chain

Two gamblers play the following game. Each period they each bet a dollar, flip a coin, and if the coin comes up heads player 1 takes the two dollars, while if it is tails player 2 does so. If one of the players goes bankrupt the game stops.

Let x, the current state of the system, be the number of dollars player 1 has at some date. The state space is $\{0, ..., d\}$ where d is the total number of dollars the two players start out with combined.

Let $P(x, y)$ be a transition function that returns the probability of next period's state being y given that the current state is x.

It is clear that

$$\begin{aligned} \sum_{y=0}^{d} yP(x,y) &= x \\ x &= 0, ..., d. \end{aligned}$$

Thus the stochastic process for x follows a martingale. (if it is not obvious to you that the process for x satisfies the above equation, try a few values for x including $x = 0$ and $x = d$, and check it)

One interesting implication regarding finite state Markov processes that are also martingales is that the extreme elements in the state space are absorbing states. For example, in the example above, if $x = 0$, then it must be the case that $P(0, y) = 0$ for all $y \neq 0$; ie if state 0 is ever reached, the system will stay in that state with probability 1.

Using martingale convergence results for the consumption / savings problem. Returning to our consumption / savings problem, the Euler inequality is

$$u'(c_t) \geq \beta(1+r)E_t u'(c_{t+1}) \qquad = \text{ if } a_{t+1} > -\phi$$

What is the limiting behavior of consumption implied by this equation? In this case we won't make any specific assumptions on the third derivative of the utility function, or the process for shocks.

At first sight this does not look like a martingale, unless we assume that $\beta(1+r)$

But we can rewrite it as a martingale for the general case ($\beta(1+r) \neq 1$) by making a change of variables

Define

$$M_t = \beta^t(1+r)^t u'(c_t) \geq 0$$

This implies that

$$M_{t+1} - M_t = \beta^t(1+r)^t \left[\beta(1+r)u'(c_{t+1}) - u'(c_t)\right]$$

The Euler equation can be rewritten as

$$E_t \left[M_{t+1} - M_t\right] \leq 0$$

which says that M_t is a supermartingale. It is also non-negative because the marginal utility of optimal consumption cannot be negative. This means (by Doob's Theorem) that M_t converges almost surely to a non-negative value.

What does this tell us about the limiting behavior of consumption?

Consider the case $\beta(1+r) > 1$. Now if M_t is converging to something, this implies that $u'(c_t)$ must be converging to zero, and therefore (if u is unbounded) that c_t is converging to infinity. (Given the borrowing constraint, this implies that asset holdings must be converging to infinity too)

Now consider $\beta(1+r) < 1$. In this case M_t might converge to zero even if $u'(c_t)$ does not converge to anything (and remains a finite randomly fluctuating variable). This is, in fact, a property of the solution.

Finally consider the case $\beta(1+r) = 1$. Chamberlain and Wilson (2000) show that c_t must also converge to infinity in this case. I will now sketch some intuition for this result.

What $\beta(1+r) = 1$ the first order condition for the household problem can equivalently be written as

$$E_t\left[u_{c,t+1} - u_{c,t}\right] \leq 0$$

We will assume that $u_{c,t}$ converges to a strictly positive limit, and derive a contradiction.

If $u_{c,t}$ converges to a strictly positive limit, then c_t must converge to a finite positive value. But each period the endowment / labor income is stochastic. The only way consumption could converge to a finite positive value would be if consumption were less than or equal to the lowest possible present value of total lifetime resources (for larger values for consumption, there would be a chance that the borrowing constraint would bind at some future date, requiring consumption to adjust at that date). But this strategy requires accumulating assets every time $e_t > e_1$,and it cannot be optimal to endlessly accumulate assets without every consuming out of them.

5.3. Approach 3: the case of i.i.d shocks. If productivity shocks are i.i.d we can take a transformation of the state variables to eliminate e as a state variable. In this case we will prove that if $\beta(1+r) \geq 1$ there is no upper bound on asset holding. This goes some way towards showing that consumption does not converge to a finite constant, and it will be useful later on when we want to construct an equilibrium with lots of agents subject to idiosyncratic risk.

Let

$$z = we + (1+r)a + \phi$$

Thus z denotes maximum disposable resources (maximum possible consumption given e and a, prices w and r, and the borrowing constraint ϕ). In terms of the single state variable z the household's budget set is given by

$$c + a' \leq z - \phi$$

If the probability distribution over e' is independent of e, then provided the household knows z he does not need to know e to solve his optimization problem. In other words, the household does not care whether the resources he has at his disposal come from savings he made in the previous period or from his current period endowment shock.

We can therefore rewrite the value function as follows.

$$V(z) = \max_{a'}\left[u(c) + \beta\sum_{e'}\pi(e')V(z')\right]$$

$$c = z - \phi - a'$$
$$z' = we' + (1+r)a' + \phi$$
$$a' \geq -\phi$$

Denote the decision rule that solves this problem $a'(z)$.

The envelope condition here is

$$V'(z) = u'(c) \tag{1}$$

Of course, the first order condition must be as before, since we have only changed notation (if you are not convinced just check for yourself by taking the FOC wrt a', and substituting in the envelope condition). Note now that the transition probabilities do not depend on e.

$$u'(c) \geq \beta(1+r)\sum_{e'} \pi(e')u'(c') \qquad = \text{if } a' \geq -\phi \tag{2}$$

Showing that assets are unbounded above is equivalent to showing that z is unbounded above, given that e is bounded and w, r, and ϕ are finite constants.

Note that total resources available next period z' is increasing in next period's endowment e'. Note also that savings are increasing in z - the more resources you have, the more you want to save. Total resources will remain bounded through time if there exists a z, denoted $z_{\max}$, such that for all e' and for all $z \leq z_{\max}$

$$z' = we' + (1+r)a'(z) + \phi \leq z_{\max}$$

Since savings are increasing in current resources z, and z' is increasing in e', it suffices to check that

$$we_N + (1+r)a'(z_{\max}) + \phi \leq z_{\max}$$

If we can find such a finite $z_{\max}$ then provided that we start out with $z_0 \leq z_{\max}$, disposable resources will never exceed $z_{\max}$.

Showing that $\beta(1+r) < 1$ is necessary for boundedness

Let us assume that there exists a $z_{\max}$ as defined above, and that that $\beta(1+r) \geq 1$. We will then derive a contradition. This contraction will tell us that for any possible candidate for $z_{\max}$, there is a positive probability that resources will increase beyond $z_{\max}$.

Substituting the envelope condition 1 into the first order condition 2 we get

$$V'(z) \geq \beta(1+r)\sum_{e'} \pi(e')V'(z')$$

If $z = z_{\max}$ this implies that

$$V'(z_{\max}) \geq \beta(1+r) \sum_{e'} \pi(e') V'(we' + (1+r)a'(z_{\max}) + \phi)$$

Since V is strictly concave, V' is decreasing

$$\sum_{e'} \pi(e') V'(we' + (1+r)a'(z_{\max}) + \phi) > V'(we_N + (1+r)a'(z_{\max}) + \phi) \geq V'(z_{\max})$$

Thus we have shown that

$$V'(z_{\max}) > \beta(1+r)V'(z_{\max})$$

Now if $\beta(1+r) \geq 1$ this is a contradiction, which means that there does not exist a $z_{\max}$ as defined. In other words, however large are a household's disposable resources, if $\beta(1+r) \geq 1$ then for the largest value for e household savings will be so large that asset holdings and disposable resources will increase. This suggests that z_t will diverge to $+\infty$.

Thus we have shown that $\beta(1+r) < 1$ is a necessary condition for asset holdings to remain bounded.