Consider the problem we discussed before

$$\sup_{\{c_t, a_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

such that

$$\begin{aligned} c_t + a_{t+1} &= y_t + (1+r)a_t \\ a_{t+1} &\geq -\phi \\ c_t &\geq 0 \\ & a_0 \text{ given} \end{aligned}$$

and where $y_t \in Y = \{y_1, y_2, ..., y_N\}$ follows a stochastic process defined by the transition probability matrix Π. Let us suppose that there is a unique ergodic distribution over y_t associated with Π defined by $p^* = \Pi p^*$. Assume that p^* defines the time zero probability distribution over y_0.

We can write this problem in the general form outlined by Stokey and Lucas as

$$\sup_{\{a_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t F(a_t, a_{t+1})$$

such that

$$a_{t+1} \in \Gamma(a_t, y_t) = \{a_{t+1} \in R; -\phi \leq a_{t+1} \leq y_t + (1+r)a_t\}$$

where

$$F(a_t, y_t, a_{t+1}) = u(y_t + (1+r)a_t - a_{t+1})$$

is the one-period return function.

The Recursive formulation of this problem is

$$v(a, y_i) = \max_{a'} \left\{ u(c) + \beta \sum_j \Pi_{ij} v(a', y_j) \right\}$$

subject to

$$\begin{aligned} c + a' &= y_i + (1 + r)a \\ a' &\geq -\phi \\ c &\geq 0 \end{aligned}$$

or, in general form,

$$v(a, y_i) = \max_{a' \in \Gamma(a, y_i)} \left\{ F(a, y_i, a') + \beta \sum_j \Pi_{ij} v(a', y_j) \right\}$$

Recall that in the non-stochastic case we showed that as long as the constraint set was non-empty, and in the limit lifetime utility was well-defined for any feasible plan, then

1. (SL Th 4.2) The function v^* defining the supremum for lifetime utility in the sequence problem (the 'true' problem) for different values for initial wealth satisfies the corresponding functional equation (the recursive formulation of the problem)

2. (SL Th 4.3) The converse: If we have a function v that solves the functional equation and satisfies

$$\lim_{n \to \infty} \beta^n v(x_n) = 0$$

where x_n could belong to any feasible sequence for x, then $v = v^*$.

There are a bunch of additional results that we didn't yet discuss for the non-stochastic case

3. (SL Th 4.4) Plans that are optimal in the sequence problem satisfy

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*)$$

4. (SL Th 4.5) If a feasible plan satisfies the equation above, and

$$\limsup_{t \to \infty} \beta^t v^*(x_t^*) \leq 0$$

then it is an optimal plan (this extra condition is needed to take care of examples like the one we talked about before)

These results (3) and (4) together link decisions in the sequential and recursive problems, while (1) and (2) link lifetime utilities in the two problems.

Without further assumptions, the optimal decision rule might take the form of a correspondence rather than a function. Define the optimal policy correspondence

$$G^*(x) = \{y \in \Gamma(x) : v^*(x) = F(x, y) + \beta v^*(y)\}$$

From (3) every optimal plan is generated from G^*, and from (4) any plan generated from G^* that satisfies the limit condition is an optimal plan.

To characterize v^* and G^* more fully, some extra assumptions are required. In general the value function will inherit properties we assume about the return function F.

Define the operator T on the space of continuous bounded functions, $C(X)$ by

$$Tf(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\}$$

5. (SL Th 4.6) Suppose that X (the set of possible values for x) is a convex subset of R^l, and the correspondence $\Gamma : X \to X$ is non-empty, compact valued and continuous

   Suppose F is bounded and $\beta < 1$.

   Then T has a unique fixed point $v \in C(X)$, and the associated policy correspondence $G : X \to X$ defined above is compact-valued and uhc. From our previous result (2) the unique fixed point v must be the supremum function for the associated sequence problem. Proving this theorem (5) relies on showing that given the stated assumptions, the Contraction Mapping Theorem applies.

6. (SL Th 4.7) Suppose that for each y, $F(., y)$ is strictly increasing and that Γ is monotone in the sense that $x \leq x'$ implies $\Gamma(x) \subseteq \Gamma(x')$. Then v is strictly increasing

7. (SL Th 4.8) Suppose that F is strictly concave, and Γ is convex. Then v is strictly concave, and G is a unique single-valued function

8. (SL Th 4.11) Suppose that F is continuously differentiable. Then v is continuously differentiable

All of these results are proved in Stokey and Lucas for a non-stochastic economy. To study the stochastic economy to the level of rigor in SL would require a heavy dose of measure theory, which we will skip for now.

Under analogous assumptions to the non-stochastic case, something like the Principle of Optimality applies. In particular, results analogous to (2) and (4) apply, so we can be sure that under certain conditions a solution v to the functional equation is the supremum function for the sequence problem, and plans generated by the correspondence G associated with the solution v attain the supremum. So we can still pursue recursive solution techniques.

Then we can procede to make further assumptions and to more tightly characterize v and G exactly as in the non-stochastic case. All of the previous results will go through pretty much unchanged, as long as the transition function for the exogenous shocks satisfies something called the Feller property. These results, analogous to the results for the non-stochastic case above, are laid out in a series of theorems in chapter 9 of Lucas and Stokey.

1 Non-stochastic version of the model revisited

Let us revisit the non-stochastic version of the model to cover two issues we skipped over last time

1. The natural borrowing constraint (previously we focussed on $\phi = 0$)

2. Optimal consumption dynamics when $\beta(1 + r) \neq 1$ (previously we only covered $\beta(1 + r) = 1$)

2 Non-stochastic version of the model revisited

Impose $c_t \geq 0$ and iterate the budget constraint forwards

$$c_t \geq 0 \Rightarrow y_t + (1+r)a_t - a_{t+1} \geq 0$$

$$\Rightarrow a_t \geq \frac{a_{t+1} - y_t}{(1+r)}$$

$$a_t \geq \frac{\frac{a_{t+2} - y_{t+1}}{(1+r)} - y_t}{(1+r)}$$

$$a_t \geq -\frac{1}{(1+r)} \sum_{j=0}^{\infty} y_{t+j} (1+r)^{-j} = -\sum_{j=1}^{\infty} y_{t+j-1} (1+r)^{-j}$$

The constraint is more naturally expressed as a limit on a_{t+1}, so updating one period gives

$$a_{t+1} \geq -\sum_{j=1}^{\infty} y_{t+j}(1+r)^{-j}$$

The nice thing about the natural borrowing constraint is that it will never bind - if it were to bind at some date t, by construction consumption would be zero at all dates $\tau \geq t+1$ (in the stochastic version, consumption would be zero with positive probability at all future dates).

Given the natural borrowing constraint, the inter-temporal first order condition (Euler equation) is

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

and the lifetime (Arrow Debreu) budget constraint is

$$
\begin{aligned}
a_0(1+r) + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} &= \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \\
a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^{t+1}} &= \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^{t+1}}
\end{aligned}
$$

The Euler equation and the lifetime budget constraint completely characterize the solution.

Call the LHS Y. Suppose (for example)

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

We can now solve for a decision rule for consumption of the form

$$c = f(Y, r, \gamma)$$

We have

$$\begin{aligned} c_t^{-\gamma} &= \beta(1+r)c_{t+1}^{-\gamma} \\ c_{t+1} &= (\beta(1+r))^{\frac{1}{\gamma}} c_t \end{aligned}$$

so

$$c_t = (\beta(1+r))^{\frac{t}{\gamma}} c_0$$

We see immediately that whether consumption is increasing or declining over time depends on whether $\beta(1+r) > 1$ or whether $\beta(1+r) < 1$. In the case $\beta(1+r) = 1$, consumption is constant (we did that case before). The rate at which consumption increases or declines over time will depend on both the magnitude of $\beta(1+r)$, and on the parameter γ

Why does γ play a role?

We can solve for c_0 (and thus for c_t) by using the lifetime budget constraint

Suppose, for example, that $\beta(1+r) = 1$. Then

$$\begin{aligned} Y &= \sum_{t=0}^{\infty} \frac{c_0}{(1+r)^{t+1}} \\ \frac{c_0}{1 - \frac{1}{1+r}} &= (1+r)Y \\ c_0 &= rY \end{aligned}$$

This is a familiar statement of the simplest version of the Permanent Income Hypothesis

A slightly weaker version of the Permanent Income / Life-Cycle Hypothesis is that consumption should not respond to predictable changes in income (even if it is not constant). This version of the PILCH relies only on ruling out the possibility of binding borrowing constraints (and does not rely on any specific assumptions on β, r or γ).