General Equilibrium with Incomplete Markets

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1. HETEROGENEITY

Consider an economy with a continuum of agents of total mass equal to 1. Each agent's productivity process is independent of all other agents current and past endowments and defined by a first order Markov process. How should we describe the equilibrium distribution of households across asset holdings and endowments? We should use a probability measure.

- Let $a_{\max}$ denote the endogenous maxmum asset holdings in equilibrium. (In the iid shock case described above $a_{\max} = a'(z_{\max})$).
- Let ψ be a probability measure defined on (S, β_S) where $S = [-\phi, a_{\max}] \times E$ and β_S is the Borel σ algebra (an appropriate set of subsets of S). Thus for any set $B \in \beta_S$, $\psi(B)$ is the mass of agents whose individual state vectors lie in B.
- Let $P((a, e), B)$ be the probability than an agent with state (a, e) today has in individual state in the set $B \in \beta_S$ tomorrow. Formally the transition function $P : S \times \beta_S \to [0, 1]$ is given by

$$P((a,e), B) = \sum_{e'} I_{(a'(a,e),e') \in B} \times \pi(e'|e)$$

Since we are in an economy with constant prices we would hope that ψ will be unchanged over time. A probability measure ψ is stationary provided that

$$\psi'(B) = \int P(((a,e), B)\, d\psi = \psi(B) \qquad \text{for all } B \in \beta_S$$

Let $x = (a, e)$ be the individual state vector. Let q be the price of a non-contingent bond that pays one unit of consumption in the next period. Given a constant q, the individual's problem, in recursive form is characterized by the following functional equation:

$$\begin{aligned} v(x;q) &= \max_{\{c, a' \in \Gamma(x;q)\}} \left\{ u(c) + \beta \sum_{e'} \pi(e'|e) v(a', e'; q) \right\} \\ \Gamma(x;q) &= \{(c, a') : c + qa' \leq a + e; c \geq 0; a' \geq -\phi\} \end{aligned}$$

Thanks to the Principle of Optimality, a solution to this functional equation (assuming it exists) is the optimal value function.

1.1. Equilibrium Definition. A stationary equilibrium is a $c(x)$, $a'(x)$, q and ψ satisfying

1. $c(x)$ and $a'(x)$ are optimal given q
2. Markets clear. In Huggett's (1993) economy this means

$$\begin{aligned} \int_S c(x)d\psi &= \int_S e(x)d\psi \\ \int_S a'(x)d\psi &= 0 \end{aligned}$$

3. ψ is a stationary probability measure

In Aiyagari's 1994 and 1995 papers and Huggett's 1997 paper the aggregate supply of assets is endogenous and derived from a production technology. This is an important difference with respect to Huggett 1993 paper in which assets are assumed to be in zero net supply. Aiyagari and McGrattan 1998 allow two sources of positive asset supply: capital used in production and government debt. These alternative models change the market clearing conditions, but do not affect the form of the individual consumption-savings problem.

1.2. Existence of a stationary measure. Under what conditions will there exist a unique stationary probability measure ψ and how can we compute this measure?

Define an order $\geq$ on S as follows. For $s, s^* \in S$ where $s = (a, e)$

$$s \geq s^* \ iff \ \begin{bmatrix} (a \geq a^* \text{ and } e = e^*) \text{ or} \\ (s^* = c = (-\phi, e_{\min})) \text{ or} \\ (s = d = (a_{\max}, e_{\max})) \end{bmatrix}$$

This is a closed order with minimum (c) and maximum (d) elements.

1.3. Theorem 2, Hopenhayn and Prescott 1987. If

- S is a compact metric space
- $\geq$ is a closed order defined on S
- (S, β_S) is a measurable space and β_S is the Borel σ algebra
- P is a transition function defined on $S \times \beta_S$
- P is increasing

- the monotone mixing condition (MMC) is satisfied - i.e. there exists $\widehat{s} \in S$, $\varepsilon > 0$, and N such that

$$P^N\left((a_{\max}, e_{\max}), \{s : s \leq \widehat{s}\}\right) > \varepsilon \text{ and } P^N\left((-\phi, e_{\min}), \{s : s \geq \widehat{s}\}\right) > \varepsilon$$

then there exists a unique stationary ψ, and for any ψ_0 this unique ψ may be found by successively applying the mapping W where

$$(W\psi)(B) = \int_S P(((a,e), B)\, d\psi \qquad \text{for } B \in \beta_S$$

1.4. Theorem 2, Huggett 1993. Obviously we would like to apply the Hopenhayn-Prescott theorem to our economy.

We have already demonstrated (for the iid shocks case) that if $\beta(1+r) < 1$ then S is compact.

It turns out that if $a'(a,e)$ is increasing in a for fixed e then the condition P is increasing is satisfied.

Thus it remains to show that the monotone mixing condition is satisfied. Choose

$$\widehat{s} = ((a'(-\phi, e_{\max}) + a_{\max})/2, e_h)$$

Now the MMC will be satisfied if decision rules are such that

(i) if a household starts out at $(a_{\max}, e_{\max})$ and receives a sequence of bad productivity shocks he will gradually reduce his asset holdings until he has wealth less than $(a'(-\phi, e_{\max}) + a_{\max})/2$. This will be the case so long as you always want to dissave in the worst endowment state until you hit the borrowing constraint.

(ii) if a household starts out with $(-\phi, e_{\min})$ and receives a series of good shocks he will gradually increase his asset holdings so that he has more than $(a'(-\phi, e_{\max}) + a_{\max})/2$. This will be the case if you always increase asset holdings in the best endowment state provided you have assets less than $a_{\max}$, where $a_{\max}$ is the smallest fixed point of $a'(a,e)$.

2. The aggregate demand for assets

Suppose that we have demonstrated that there exists a unique stationary probability measure ψ defined on (S, β_S) where $S = [-\phi, a_{\max}] \times E$ which behaves continuously with respect to the parameters b and q. Define $r = \frac{1-q}{q}$. Since we have shown that a necessary condition for assets to remain bounded is that $\beta(1+r) < 1$, we focus on this case. Let aggregate asset holdings given the stationary distribution ψ_r associated with a particular interest rate r (and implicitly a particular borrowing limit b) be denoted $A(r)$.

$$A(r) = \int a'(a,e;r) d\psi_r$$

The preceeding discussion goes some way towards demonstrating that as r tends to the rate of time preference (given by $\rho = (1-\beta)/\beta$) from below, the aggregate demand for assets $A(r)$ tends to infinity. The intuition is straightforward - if $r < \rho$ there is a cost to holding assets in that the return they offer does not compensate the household for the implied postponement of consumption. On the other hand, there is a benefit in that holding more assets on average allows households to smooth consumption more easily. As $r \to \rho$ the cost of holding savings tends to zero while any finite quantity of precautionary savings will still be exhausted following a sufficiently long sequence of low realizations of productivity.

For $r = 0$, define $m = a + \phi$. The constraints

$$c = e + a - a'$$

$$a' \geq -\phi$$

may be rewritten as

$$c = e + m - m'$$

$$m' \geq 0$$

This implies that if optimal savings in the original economy (with $\phi > 0$) for some particular asset level a is x, then optimal savings in an alternative economy with a borrowing limit of zero and asset level $a + \phi$ is $x + \phi$, ie

$$a'(a + \phi; 0) = \phi + a'(a; \phi)$$

Note that this is true for any a. Thus if we have a stationary measure over a and e with the borrowing constraint ϕ, then we can construct a stationary measure for the economy with a zero borrowing constraint by simply increasing everyone's assets by the amount ϕ. Thus if aggregate savings in the first economy is A, then aggregate savings in the second economy is $A + \phi$.

For r sufficiently low, all households will want to borrow the maximum permissible amount. In this case $A(r) = -\phi$.

Note that if earnings were certain (or equivalently markets were complete) $A(r) = -\phi$ for all $r < \rho$.

2.1. Huggett 1993. In this economy the single asset consists of risk-free loans offered by other households. There are no 'outside' assets. The market clearing condition is $A(r) = 0$.

2.2. Aiyagari 1994. In Aiyagari (1994) aggregate output is given by

$$Y = F(K, N)$$

Since labor supply is exogenous and by the law of large numbers, average labor productivity is constant, $N = \int e d\psi_r$ is a constant and does not depend on K. Thus we can express aggregate capital and the wage as functions of the interest rate,

$$K = K(r)$$

$$w = w(r)$$

Recall that in the description of the household's problem, r is the net return to saving. Thus if assets are interpreted as capital, then $r = MPK - \delta$.

Since capital is the only asset in positive net supply in this economy, the equilibrium r is given by the solution to

$$A(r) = K(r)$$

The only difference in terms of computing $A(r)$ is that now, for each value for r we must first compute the implied $w(r)$ and use this an input to households' savings problems to calculate $A(r)$.

If the production function is Cobb-Douglas with depreciation rate δ, $K(r)$ will be downward-sloping, tending to infinity as r tends to $-\delta$ and to zero as r tends to infinity.

It is clear from the discussion of the shapes of the $A(r)$ and $K(r)$ curves that the capital stock is higher and the interest rate is lower in the economy with idiosyncratic shocks and borrowing constraints than in the standard complete markets economy (in which $r = \rho$).

2.3. Aiyagari and McGrattan 1998. In Aiyagari and McGrattan (1998) there is a government that issues debt. This debt provides an additional savings instrument for households. Aiyagari and McGrattan assume that the government guarantees the real return on the debt that it issues. Furthermore there are no transaction costs or taxes associated with trading either shares in the aggregate capital stock or government bonds. Taken together these assumptions imply that in equilibrium the real return on capital and government debt must be the same. However, this does not imply that the steady state of the economy does not depend on the real amount of outstanding government debt

The steady state condition is now

$$A(r) = K(r) + B$$

where B is the solution to

$$rB = T - G$$

and $T - G$ is the government's primary surplus.

Aiyagari and McGrattan compare steady states across which G is held constant and B is varied. The lump-sum tax level T (or proportional tax rates in a later example) is then adjusted to ensure government budget balance at the equilibrium interest rate. The question they ask is what steady state level of debt maximizes a utilitarian social welfare function? With lump sum taxes there are two forces working in opposite directions. Firstly, debt enhances liquidity - from the graph the equilibrium interest rate increases when we increase steady state debt meaning that it is now less costly for households to hold precautionary savings and thus easier for them to smooth consumption. Secondly, government debt crowds out private capital - from the graph the increase in aggregate asset holdings is smaller than the increase in government debt - meaning that per capita consumption is reduced. An additional consideration with proportional taxes is that more debt implies higher and more distortionary tax rates.

Are there any problems with the welfare comparison across steady states Aiyagari McGrattan undertake?

3. Questions and extensions

How do we solve numerically for equilibria in these economies?

See next set of notes

How far is r from ρ?

Huggett: quarterly model, $\beta = 0.96$ (annual), CRRA preferences, $\sigma = 1.5$, $e_h = 1$, $e_l = 0.1$, each period is two months, $\pi(e_h|e_l) = 0.925$, $\pi(e_h|e_l) = 0.5$, average duration of low endowment state (unemployment) is 2 periods = 17 weeks, credit limit -6 (slightly greater than one year's average endowment). Results $r = 3.4\%$ compared to $\rho = \frac{1-\beta}{\beta} = 4.2\%$

How much inequality in consumption and weath does this model generate?

How do these results depend on the persistence, variance, and higher moments of the shock process?

How do the results depend on properties of preferences?

How do they depend on the position of the borrowing constraint?

If we start outside the stationary distribution, what does transition look like?

Can we handle aggregate shocks in a framework like this?

How close are allocations to complete markets / autarky?

What would happen if we introduced endogenous labor supply?

What would happen if we introduced heterogeneity in preferences?

Would things look much different in an OLG version of this model?