Notes on Alvarez and Jermann, "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," Econometrica 2000

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1 Model

Consider a pure exchange economy with I agents and one good

Let $\{z_t\}$ denote a Markov process with generic element $z \in Z$, and transition probabilities given by Π. Individual endowments are given by a function that depends only on $z_t : e_i(z^t) = \varepsilon_i(z_t)$.

Let $z^t = (z_0, z_1, z_2, ...z_t)$

Expected utility at z^t is given by

$$U(c_i(z^t)) = \sum_{s=t}^{\infty} \sum_{z^s \in Z^s} \beta^{s-t} \pi(z^s | z^t) u(c_i(z^s))$$

Note that

$$U(c_i(z^t)) = u(c_i(z^t)) + \beta \sum_{z^{t+1} \in Z} \pi(z^{t+1} | z^t) U(c_i(z^{t+1}))$$

Let $U(e_i(z^t))$ denote expected lifetime utility for agent i in autarky

An allocation is *resource feasible* if

$$\sum_{i=1}^{I} c_i(z^t) = E(z^t) = \sum_{i=1}^{I} e_i(z^t)$$

and it *satisfies participation constraints* if

$$U(c_i(z^t)) \geq U(e_i(z^t)) \qquad \forall t \geq 0, z^t \in Z^t, i \in I$$

An allocation $\{c_i(z^t)\}$ is *feasible* if it is resource feasible and satisfies participation constraints.

An allocation is *efficient* if it is feasible and not Pareto dominated by other feasible allocations

Suppose an allocation is efficient. Then if some some agent j

$$U(c_j(z^t, z_{t+1})) > U(e_j(z^t, z_{t+1}))$$

it is the case that

$$\frac{u'(c_j(z^t, z_{t+1}))}{u'(c_j(z^t))} = \max_{i \in I} \frac{u'(c_i(z^t, z_{t+1}))}{u'(c_i(z^t))}$$

This proposition (3.2) tells us that **all unconstrained agents have the same MRS, and all constrained agents have lower MRS's**.

The intuition for the proposition is as follows. Suppose (contrary to the proposition) that some other agent $i \neq j$ has a higher MRS. Then the planner could make i better off and j no worse off by giving j (marginally) more consumption today and promising a bit less in z_{t+1} tomorrow such that $U(c_j(z^t))$ is unchanged, and giving i less consumption today and promising more in z_{t+1} tomorrow. Would this violate j's participation constraint? No, since even if j has slightly less consumption tomorrow in z_{t+1} he will not choose to default, since by assumption there was initially some slack in his participation constraint. Why does this make i better off? Because i, by assumption, has a higher relative taste for consumption in z_{t+1}.

2 Competitive equilbrium with trade in Arrow securities

Let $q(z^t, z')$ be the price at z^t of one unit of consumption delivered if $z_{t+1} = z'$.

Let $a_i(z^t, z')$ be the holdings of agent i of this asset and the lower limit on holdings of this asset for agent i be $B_i(z_t, z')$

The individual's budget constraints are now

$$c_i(z^t) + \sum_{z' \in Z} a_i(z^t, z')q(z^t, z') \leq a_i(z^t) + e_i(z^t)$$

and

$$a_i(z^t, z') \geq B_i(z^t, z')$$

Agents are NOT allowed to default in this problem.

An *equilibrium with solvency constraints* is a set of prices $\{q(z^t)\}$ such that $\forall t, z^t$:

1. Individuals maximize expected lifetime utility subject to the sequence of budget constraints, taking as given initial wealth and the initial endowment.

2. Markets clear

$$\sum_{i \in I} c_i(z^t) = E(z^t)$$

$$\sum_{i \in I} a_i(z^t, z') = 0 \qquad \forall z'$$

Sufficient conditions for maximum in household's problem:
First order condition with respect to $a_i(z^t, z')$

$$\begin{aligned} u'(c_i(z^t))q(z^t, z') &\geq \beta\pi(z'|z^t)u'(c_i(z^t, z')) \\ &= \text{ if } a_i(z^t, z') > B_i(z^t, z') \end{aligned}$$

Transversality condition

$$\lim_{t \to \infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u'(c_i(z^t)). \left[a_i(z^t) - B_i(z^t)\right] = 0$$

Let the value function for agent i with assets a at z^t be denoted $V_i(a, z^t)$

$$V_i(a, z^t) = u(c_i(z^t)) + \beta \sum_{z'} \pi(z'|z^t) V_i(a_i(z^t, z'), \left(z^t, z'\right))$$

An equilibrium has *solvency constraints that are not too tight* if

$$V_i(B_i(z^{t+1}), z^{t+1}) = U(e_i(z^{t+1})) \qquad \forall t \geq 0,\ z^{t+1} \in Z^{t+1}$$

If solvency constraints are not too tight, then agents are able to take on as much contingent debt as possible while never wanting to default in the next period. If solvency constraints are not too tight then

$$U(c_i(z^t)) \geq U(e_i(z^t))$$

and

$$U(c_i(z^t)) = U(e_i(z^t)) \Leftrightarrow a_i(z^t) = B_i(z^t)$$

Given an allocation $\{c_i(z^t)\}$ for $i = 1, ..., I$ define

$$q(z^t, z') = \max_{i \in I} \left\{ \beta \frac{u'(c_i(z^t, z'))}{u'(c_i(z^t))} \pi(z'|z^t) \right\}$$

and

$$Q_0(z^t|z_0) = q(z_0, z_1).q_1(z_0, z_1, z_2)...q(z^{t-1}, z_t)$$

This is the price of one unit of consumption delivered in z^t in units of consumption at time zero (one period interest rates can be read off the intertemporal first order conditions of the unconstrained agents)

The *implied interest rates* for the allocation $\{c_i\}$ are *high* if

$$\sum_{t \geq 0} \sum_{z^t \in Z^t} Q_0(z^t|z_0) \left(\sum_i c_i(z^t) \right) < +\infty$$

High interest rates ensure finiteness of the value of endowment implied by a given allocation.

Now the goal is to connect efficient allocations (the type of allocations we solved for in solving the planner's problems in the Kocherlakota / Kehoe Perri model) with the equilibrium allocations from the set up described above.

Alvarez and Jermann show that **any constrained efficient allocation with high implied interest rates can be decentralized as a competitive equilbirium with solvency constraints that are not too tight** (the second welfare theorem, corollary 4.2)

The idea is that in the equilibrium set up, solvency constraints that are not too tight replace participation constraints. In the equilibrium set up an agent whose solvency constraint is not binding has the highest MRS. In the the efficient allocation an agent whose participation constraint is not binding has the highest MRS.

Note that even though the constrained efficient allocation can be decentralized, it does not follow that every competitive equilibrium with solvency constraints that are not too tight is efficient (the first welfare theorem). For example, it is possible to construct an equilibrium with solvency constraints that gives autarkic allocations:

$$\begin{aligned} c_i(z^t) &= e_i(z^t) \\ a_i(z^t, z') &= B_i(z^t, z') = 0 \\ q(z^t, z') &= \max_{i \in I} \left\{ \beta \frac{u'(e_i(z^t, z'))}{u'(e_i(z^t))} \pi(z'|z) \right\} \end{aligned}$$

In this example, no one can borrow, and no-one wants to save (the guy with the highest MRS is indifferent about saving). By construction the solvency constraints are not too tight: with $a_i(z^t, z') = 0$ the value of default is the same as the value of not defaulting (in both cases you get the endowment stream).

Alvarez and Jermann then proceed to consider the first welfare theorem. They show (corollary) that **allocations in an equilibrium with solvency constraints that are not too tight and that have high implied interest rates and that satisfy an additional technical condition (see below) are constrained efficient** (note that in the autarky example above we did not check the high implied interest rates condition).

This part of the paper is a bit tricky, since their strategy is to first show that allocations in their equilibrium with solvency constraints look like allocations in a different type of equilbirium developed by Kehoe and Levine in which individuals face participation constraints rather than solvency constraints. They are able to show equivalence between the two equilbrium concepts as long as implied interest rates are high, solvency constraints are not too tight and the following technical condition is satisfied:

For each $i \in I$ there is a constant ζ_i such that for all t, z^t

$$\left|u(c_i(z^t)\right| \leq \zeta_i.u'(c_i(z^t)).c_i(z^t)$$

This condition is hard to interpret, but it is satisfied if RRA $\neq 1$ at zero consumption, if $u'(0) < \infty$, or if consumption is uniformly bounded away from zero.

The problem in the Kehoe Levine economy is

$$\max_{c_i(z^t)} U(c_i(z^t))$$

subject to

$$\sum_{t \geq 0} \sum_{z^t \in Z^t} \left(c_i(z^t) - e_i(z^t)\right) Q_0(z^t|z_0) \leq a_{i,0}$$

where the $Q_0(z^t|z_0)$ terms are date zero Arrow-Debreu prices, and

$$U(c_i(z^t)) \geq U(e_i(z^t)) \qquad \forall t \geq 0, z^t \in Z^t$$

Allocations in the Kehoe Levine equilibrium are efficient, since it is a standard Arrow-Debreu equilibrium. Thus if allocations in the equilibrium with solvency constraints are equivalent to those in the KL equilbrium, we have shown the first welfare theorem.

3 Some More Results

Autarky is the only feasible allocation when people are very impatient, people are not very risk averse, shocks are not very volatile, and shocks are very persistent (Proposition 4.9). The proof of these results involves showing that as the discount factor tends to zero, risk aversion tends to zero, the variance of shocks tends to zero, or the persistence of shocks tends to one, the autarky value of the endowment converges to a finite limit - implied interest rates are high. Because implied interest rates are high, the conditions for the first welfare theorem apply, and the autarky equilibrium with solvency constraints is constrained efficient.

Given that the shock process is Markov, in autarky values for previous shocks are irrelevant for prices. Let $A(z)$ denote the value of the aggregate endowment given current shock z.

$$A(z) = \sum_{z' \in Z} q(z, z') \left[e(z') + A(z')\right]$$

where

$$e(z') = \sum_{i \in I} e_i(z')$$

and

$$q(z, z') = \beta \max_{i \in I} \left(\frac{u'(e_i(z'))}{u'(e_i(z))}\right) \pi(z'|z)$$

Substituting in the expression for prices

$$A(z) = \beta \sum_{z' \in Z} [e(z') + A(z')] \max_{i \in I} \left(\frac{u'(e_i(z'))}{u'(e_i(z))} \right) \pi(z'|z)$$

Lets define new fake transition probabilities that add to one, and a fake discount rate $\beta^*(z)$:

$$\begin{aligned} \pi^*(z'|z) &= \frac{\max_{i \in I} \left(\frac{u'(e_i(z'))}{u'(e_i(z))} \right) \pi(z'|z)}{\sum\limits_{z' \in Z} \max_{i \in I} \left(\frac{u'(e_i(z'))}{u'(e_i(z))} \right) \pi(z'|z)} \\ \beta^*(z) &= \beta \sum_{z' \in Z} \max_{i \in I} \left(\frac{u'(e_i(z'))}{u'(e_i(z))} \right) \pi(z'|z) \end{aligned}$$

In terms of these new variables

$$A(z) = \beta^*(z) \sum_{z' \in Z} [e(z') + A(z')] \, \pi^*(z'|z)$$

If we can show that $\beta^*(z) < 1$ for all z, then the present value of the aggregate endowment must be finite. It is straightforward to verify that the $\beta^*(z) < 1$ for all z condition is in fact met in all the limiting cases described above.

Further (Unsurprising) Results: (1) if some risk sharing is possible in a constrained efficient allocation, then it has high implied interest rates, (2) in an equilibrium with solvency constraints that are not too tight, and where the implied interest rates are high, the solvency constraints are negative

4 Asset Prices

In the equilbirium with solvency constraints, the market clearing price of a one period Arrow security paying off in a particular state next period is the state contingent MRS of the individual(s) with the highest MRS. All other individuals would like to sell short this particular Arrow security, but they are limited in the extent to which they can do this by the asset-specific borrowing constraint.

Given the prices of Arrow securities it is possible to price all other possible assets. The prices of these assets will typically be higher than in an economy without solvency constraints, since demand for assets is bounded below by the solvency constraints.