Notes on Burstein and Monge*

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1 Relative Firm Size

Profit for local firm e_j in country i is

$$\pi_{i,j} = z_i x_i e_j (k_j^{\alpha} n_j^{1-\alpha})^v - w_i n_j - r_i k_j$$

Profit for foreign firm e_k in country i is

$$\pi_{i,k} = z_i x_1 e_k (k_k^{\alpha} n_k^{1-\alpha})^v - w_i n_k - r_i k_k$$

Take first order conditions wrt capital and labor:
We get

$$\begin{aligned} w_i &= z_i x_i e_j v (k_j^{\alpha} n_j^{1-\alpha})^{v-1} (1-\alpha) k_j^{\alpha} n_j^{-\alpha} = z_i x_1 e_k v (k_k^{\alpha} n_k^{1-\alpha})^{v-1} (1-\alpha) k_k^{\alpha} n_k^{-\alpha} \\ r_i &= z_i x_i e_j v (k_j^{\alpha} n_j^{1-\alpha})^{v-1} \alpha k_j^{\alpha-1} n_j^{1-\alpha} = z_i x_1 e_k v (k_k^{\alpha} n_k^{1-\alpha})^{v-1} \alpha k_k^{\alpha-1} n_k^{1-\alpha} \end{aligned}$$

Dividing through

$$\frac{w_i}{r_i} = \frac{1-\alpha}{\alpha} \left(\frac{k_j}{n_j} \right) = \frac{1-\alpha}{\alpha} \left(\frac{k_k}{n_k} \right)$$

so all firms irrespective of individual productivity or location have the same capital to labor ratio.

Also

$$\frac{x_i e_j}{x_1 e_k} = \left(\frac{n_j}{n_k} \right)^{1-v}$$

$$\left(\frac{x^i e_j}{x^1 e_k} \right)^{\frac{1}{1-v}} = \left(\frac{n_j}{n_k} \right)$$

This expression determines the relative size of firms.

To determine the absolute size of firms use the labor market clearing condition:

$$L_i \int_{\overline{e}^i}^{\infty} n_j dF(e) + L_1 m^i \int_{\overline{e}^1}^{\infty} n_k dF(e) = F(\overline{e}^i) L_i$$

*All these notes contain are some algebraic derivations for the equations in the Burstein-Monge paper.

Imagine a firm with $x^1 e_k = 1$. This firm employs $n_k = n^*$ workers. For any other firm

$$n_j = \left(x^i e_j\right)^{\frac{1}{1-v}} n^*$$

Substitute this expression for n into the labor market clearing condition:

$$L_i \int_{\overline{e}^i}^{\infty} \left(x^i e_j\right)^{\frac{1}{1-v}} n^* dF(e) + L_1 m^i \int_{\overline{e}^1}^{\infty} \left(x^1 e_j\right)^{\frac{1}{1-v}} n^* dF(e) = F(\overline{e}^i) L_i$$

So for firm j,

$$n_j = \left(x^i e_j\right)^{\frac{1}{1-v}} \frac{F(\overline{e}^i)}{\left(x^{i\frac{1}{1-v}} \int_{\overline{e}^i}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e) + \frac{L_1}{L_i} m^i x^{1\frac{1}{1-v}} \int_{\overline{e}^1}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e)\right)}$$

Firm $j's$ share of aggregate labor, capital and output is

$$s_j = \frac{n_j}{N_i} = \frac{n_j}{F(\overline{e}^i) L_i} = \left(x^i e_j\right)^{\frac{1}{1-v}} \frac{1}{L_i \left(x^{i\frac{1}{1-v}} \int_{\overline{e}^i}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e) + \frac{L_1}{L_i} m^i x^{1\frac{1}{1-v}} \int_{\overline{e}^1}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e)\right)}$$

(we can divide by this to go from individual to aggregate output)

2 Output and Factor Prices

So output for firm j is

$$\begin{aligned}
y_{i,j} &= z_i x_i e_j (k_j^{\alpha} n_j^{1-\alpha})^v = z_i x_i e_j n_j^v \left(\frac{K}{N}\right)^{\alpha v} \\
&= z_i x_i e_j \left(\left(x^i e_j\right)^{\frac{1}{1-v}} \frac{F(\overline{e}^i)}{\left(x^{i\frac{1}{1-v}} \int_{\overline{e}^i}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e) + \frac{L_1}{L_i} m^i x^{1\frac{1}{1-v}} \int_{\overline{e}^1}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e)\right)} \right)^v \left(\frac{K}{N}\right)^{\alpha v} \\
&= z_i x_i^{\frac{1}{1-v}} e_j^{\frac{1}{1-v}} F(\overline{e}^i)^{v(1-\alpha)} \left(L_i x^{i\frac{1}{1-v}} \zeta^i + L_1 m^i x^{1\frac{1}{1-v}} \zeta^1\right)^{-v} K^{\alpha v} L_i^{v(1-\alpha)}
\end{aligned}$$

where $\zeta^i = \int_{\overline{e}^i}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e)$.

So aggregate output in country i is (eq 3.1):

$$Y_i = y_{i,j}/s_j = z_i F(\overline{e}^i)^{v(1-\alpha)} \left(L_i x^{i\frac{1}{1-v}} \zeta^i + L_1 m^i x^{1\frac{1}{1-v}} \zeta^1\right)^{1-v} K^{\alpha v} L_i^{v(1-\alpha)}$$

Equilibrium wage in country i (eq 3.2):

$$\begin{aligned}
w_i &= \frac{dy_{i,j}}{dn_j} = v(1-\alpha) z_i x_i e_j n_j^{v(1-\alpha)-1} k^{\alpha v} = v(1-\alpha) z_i x_i e_j n_j^{v-1} \left(\frac{K}{N}\right)^{\alpha v} \\
&= v(1-\alpha) z_i x_i e_j \left((x_i e_j)^{\frac{1}{1-v}} \frac{F(\overline{e}^i)}{\left(x_i^{\frac{1}{1-v}} \int_{\overline{e}^i}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e) + \frac{L_1}{L_i} m^i x_1^{\frac{1}{1-v}} \int_{\overline{e}^1}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e)\right)} \right)^{v-1} \left(\frac{K}{N}\right)^{\alpha v} \\
&= v(1-\alpha) z_i \left(x_i^{\frac{1}{1-v}} \int_{\overline{e}^i}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e) + \frac{L_1}{L_i} m^i x_1^{\frac{1}{1-v}} \int_{\overline{e}^1}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e) \right)^{1-v} \left(\frac{K}{L_i}\right)^{\alpha v} F(\overline{e}^i)^{v(1-\alpha)-1}
\end{aligned}$$

Note that if $m_i = 0$ then

$$w_i = x_i v(1-\alpha)\left(\zeta^i\right)^{1-v} z_i \left(\frac{K_i}{L_i}\right)^{\alpha v} F(\overline{e}^i)^{v(1-\alpha)-1}$$

Similarly, the interest rate is

$$\begin{aligned}
r_i &= v\alpha z_i x_i e_j n_j^{v(1-\alpha)} k^{\alpha v-1} = v\alpha z_i x_i e_j n_j^{v-1}\left(\frac{K}{N}\right)^{\alpha v-1} \\
&= v\alpha z_i \left(x^{i\frac{1}{1-v}} \int_{\overline{e}^i}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e) + \frac{L_1}{L_i} m^i x^{1\frac{1}{1-v}} \int_{\overline{e}^1}^{\infty} (e_j)^{\frac{1}{1-v}} dF(e)\right)^{1-v} \left(\frac{K}{L_i}\right)^{\alpha v-1} F(\overline{e}^i)^{v(1-\alpha)}
\end{aligned}$$

and if $m_i = 0$ this simplifies to

$$r_i = \alpha v \left(x^{i\frac{1}{1-v}}\zeta^i\right)^{1-v} z_i K_i^{(\alpha v-1)} F(\overline{e}^i)^{v(1-\alpha)} L_i^{1-\alpha v}$$

Profits for firm j are

$$\begin{aligned}
\pi_j &= z_i x_i e_j n_j^v \left(\frac{K_i}{F(e)L_i}\right)^{va} - r_i k_j - w_i n_j \\
&= z_i x_i e_j n_j^v \left(\frac{K_i}{F(e)L_i}\right)^{va} - r_i \left(\frac{K_i}{F(e)L_i}\right) n_j - w_i n_j \\
&= z_i x_i e_j n_j^v \left(\frac{K_i}{L_i}\right)^{va} F(e)^{-v\alpha} - \left(v z_i \left(x^{i\frac{1}{1-v}}\zeta^i + \frac{L_1}{L_i} m^i x^{1\frac{1}{1-v}}\zeta^1\right)^{1-v} \left(\frac{K}{L_i}\right)^{\alpha v} F(\overline{e}^i)^{v(1-\alpha)-1}\right) n_j
\end{aligned}$$

Recall that

$$n_j = \left(x^i e_j\right)^{\frac{1}{1-v}} \frac{F(\overline{e}^i)}{\left(x^{i\frac{1}{1-v}}\zeta^i + \frac{L_1}{L_i} m^i x^{1\frac{1}{1-v}}\zeta^1\right)} = \left(x^i e_j\right)^{\frac{1}{1-v}} \frac{F(\overline{e}^i)}{\kappa}$$

So profits are (eq 3.3):

$$\begin{aligned}
\pi_j &= z_i x_i e_j \left(\left(x^i e_j\right)^{\frac{1}{1-v}} \frac{F(\overline{e}^i)}{\kappa}\right)^v \left(\frac{K_i}{L_i}\right)^{va} F(e)^{-v\alpha} - \left(v z_i \kappa^{1-v} \left(\frac{K}{L_i}\right)^{\alpha v} F(\overline{e}^i)^{v(1-\alpha)-1}\right) (x_i e_j)^{\frac{1}{1-v}} \frac{F(\overline{e}^i)}{\kappa} \\
&= (1-v) z_i x_i^{\frac{1}{1-v}} \kappa^{-v} \left(\frac{K_i}{L_i}\right)^{va} F(e)^{v(1-\alpha)} e_j^{\frac{1}{1-v}}
\end{aligned}$$

3 Expressions for Country 1

Expressions are are slightly different in country 1, since the labor market clearing condition is different

$$n_j = (e_j)^{\frac{1}{1-v}} n^*$$

$$L_1 \left(1 - \sum_{i \neq 1} m_i\right) \zeta^1 n^* = F(\overline{e}^1) L_1$$

$$n^* = \frac{F(\overline{e}^1)}{\zeta^1 \left(1 - \sum_{i \neq 1} m_i\right)}$$

$$n_j = (e_j)^{\frac{1}{1-v}} \frac{F(\overline{e}^1)}{\zeta^1} = (e_j)^{\frac{1}{1-v}} \frac{F(\overline{e}^1)}{\zeta^1 \left(1 - \sum_{i \neq 1} m_i\right)}$$

Output in country 1 of firm j is

$$\begin{aligned}
z_1 x_1 e_j (k_j^{\alpha} n_j^{1-\alpha})^v &= z_1 x_1 e_j n_j^v \left(\frac{K}{N}\right)^{\alpha v} \\
&= z_1 x_1 e_j \left((e_j)^{\frac{1}{1-v}} \frac{F(\overline{e}^1)}{\zeta^1 \left(1 - \sum_{i \neq 1} m_i\right)} \right)^v \left(\frac{K}{N}\right)^{\alpha v} \\
&= z_1 x_1 e_j^{\frac{1}{1-v}} \left(\zeta^1 \left(1 - \sum_{i \neq 1} m_i\right)\right)^{-v} \left(\frac{K_1}{L_1}\right)^{\alpha v} F(\overline{e}^1)^{v(1-\alpha)}
\end{aligned}$$

$$Y_1 = z_1 x_1 \left(\zeta^1 \left(1 - \sum_{i \neq 1} m_i\right)\right)^{1-v} \left(\frac{K_1}{L_1}\right)^{\alpha v} F(\overline{e}^1)^{v(1-\alpha)} L_1$$

Computing w r and π as before:

$$\begin{aligned}
w_1 &= v(1-\alpha) z_1 x_1 e_j n_j^{v(1-\alpha)-1} k^{\alpha v} = v(1-\alpha) z_1 x_1 e_j n_j^{v-1} \left(\frac{K_1}{N_1}\right)^{\alpha v} \\
&= v(1-\alpha) z_1 x_1 e_j \left((e_j)^{\frac{1}{1-v}} \frac{F(\overline{e}^1)}{\zeta^1 \left(1 - \sum_{i \neq 1} m_i\right)} \right)^{v-1} \left(\frac{K_1}{N_1}\right)^{\alpha v} \\
&= v(1-\alpha) z_1 x_1 \left(\zeta^1\right)^{1-v} \left(1 - \sum_{i \neq 1} m_i\right)^{1-v} \left(\frac{K_1}{L_1}\right)^{\alpha v} F(\overline{e}^1)^{v-1-\alpha v}
\end{aligned}$$

$$r_1 = v\alpha z_1 x_1 \left(\zeta^1\right)^{1-v} \left(1 - \sum_{i\neq 1} m_i\right)^{1-v} \left(\frac{K_1}{L_1}\right)^{\alpha v - 1} F(\overline{e}^1)^{v-\alpha v}$$

$$\begin{aligned}
\pi_{1,j} &= z_1 x_1 e_j n_j^v \left(\frac{K_1}{N_1}\right)^{\alpha v} - z_1 x_1 v \left(\zeta^1\right)^{1-v} \left(1 - \sum_{i\neq 1} m_i\right)^{1-v} \left(\frac{K_1}{L_1}\right)^{\alpha v} F(\overline{e}^1)^{v-1-\alpha v} n_j \\
&= z_1 x_1 e_j \left((e_j)^{\frac{1}{1-v}} \frac{F(\overline{e}^1)}{\zeta^1 \left(1 - \sum_{i\neq 1} m_i\right)} \right)^v \left(\frac{K_1}{L_1}\right)^{\alpha v} F(\overline{e}^1)^{-\alpha v} \\
&\quad - v z_1 x_1 \left(\zeta^1\right)^{1-v} \left(1 - \sum_{i\neq 1} m_i\right)^{1-v} \left(\frac{K_1}{L_1}\right)^{\alpha v} F(\overline{e}^1)^{v-1-\alpha v} (e_j)^{\frac{1}{1-v}} \frac{F(\overline{e}^1)}{\zeta^1 \left(1 - \sum_{i\neq 1} m_i\right)} \\
&= (1-v) z_1 x_1 e_j^{\frac{1}{1-v}} \left(\zeta^1 \left(1 - \sum_{i\neq 1} m_i\right)\right)^{-v} \left(\frac{K_1}{L_1}\right)^{\alpha v} F(\overline{e}^1)^{v-\alpha v}
\end{aligned}$$

(this is eq 3.4 when $e_j = 1$)

4 Comparing Profits at Home (1) and Abroad (i)

Suppose now the same firm operated in country i. Adapting (3.3)

$$\pi_{i,j} = (1-v) z_i \left(x_1\right)^{\frac{1}{1-v}} \kappa^{-v} \left(\frac{K_i}{L_i}\right)^{va} F(e)^{v(1-\alpha)} e_j^{\frac{1}{1-v}}$$

Now if there are no foreign firms in foreign markets

$$\kappa = \left(x^{i\,\frac{1}{1-v}} \zeta^i + \frac{L_1}{L_i} m^i x^{1\,\frac{1}{1-v}} \zeta^1\right) = x^{i\,\frac{1}{1-v}} \zeta^i$$

so

$$\begin{aligned}
\pi_{i,j} &= (1-v) z_i \left(x_1\right)^{\frac{1}{1-v}} \left(x^{i\,\frac{1}{1-v}} \zeta^i\right)^{-v} \left(\frac{K_i}{L_i}\right)^{va} F(e)^{v(1-\alpha)} e_j^{\frac{1}{1-v}} \\
&= (1-v) z_i x_1^{\frac{1}{1-v}} x_i^{\frac{-v}{1-v}} \zeta^{i-v} \left(\frac{K_i}{L_i}\right)^{va} F(e)^{v(1-\alpha)} e_j^{\frac{1}{1-v}}
\end{aligned}$$

So the ratio of after-tax profits from operating abroad to at home when no other firms do is:

$$\frac{(1-\tau_F^i)(1-v)z_i x_1^{\frac{1}{1-v}} x_i^{\frac{-v}{1-v}} \zeta^{i-v} \left(\frac{K_i}{L_i}\right)^{va} F(\overline{e}^i)^{v(1-\alpha)} e_j^{\frac{1}{1-v}}}{(1-\tau_D^1)(1-v)z_1 x_1 e_j^{\frac{1}{1-v}} \left(\zeta^1\left(1-\sum_{i\neq 1} m_i\right)\right)^{-v} \left(\frac{K}{L_1}\right)^{\alpha v} F(\overline{e}^1)^{v-\alpha v}}$$

$$= \frac{(1-\tau_F^i)z_i x_1^{\frac{v}{1-v}} \left(\zeta^1\right)^v \left(\frac{K_i}{L_i}\right)^{va} F(\overline{e}^i)^{v(1-\alpha)}}{(1-\tau_D^1)z_1 x_i^{\frac{v}{1-v}} \left(\zeta^i\right)^v \left(\frac{K_1}{L_1}\right)^{\alpha v} F(\overline{e}^1)^{v(1-\alpha)}}$$

Take the numerator and denominator to the power $\frac{1}{v}$ to get eq 3.6..

On a balanced growth path when the return to capital is equalized

$$r_i = \alpha v \left(x_i^{\frac{1}{1-v}} \zeta^i\right)^{1-v} z_i K_i^{(\alpha v-1)} F(\overline{e}^i)^{v(1-\alpha)} L_i^{1-\alpha v}$$

$$r_1 = v\alpha z_1 x_1 \left(\zeta^1\right)^{1-v} \left(\frac{K_1}{L_1}\right)^{\alpha v-1} F(\overline{e}^1)^{v(1-\alpha)}$$

If interest rates are equalized (assuming equal tax rates $\tau_F^i = \tau_D^1$ and threshold managerial skill $\overline{e}^1 = \overline{e}^i$)

$$z_i x_i \left(\zeta^i\right)^{1-v} \left(\frac{K_i}{L_i}\right)^{\alpha v-1} = z_1 x_1 \left(\zeta^1\right)^{1-v} \left(\frac{K_1}{L_1}\right)^{\alpha v-1}$$

so

$$\frac{\left(\frac{K_i}{L_i}\right)^{va}}{\left(\frac{K_1}{L_1}\right)^{v\alpha}} = \left(\frac{z_1 x_1 \left(\zeta^1\right)^{1-v}}{z_i x_i \left(\zeta^i\right)^{1-v}}\right)^{\frac{\alpha v}{\alpha v-1}}$$

so the ratio of after-tax profits discussed above is

$$\frac{z_i x_1^{\frac{v}{1-v}} \left(\zeta^1\right)^v \left(\frac{K_i}{L_i}\right)^{va}}{z_1 x_i^{\frac{v}{1-v}} \left(\zeta^i\right)^v \left(\frac{K}{L_1}\right)^{\alpha v}} = \frac{z_i x_1^{\frac{v}{1-v}} \left(\zeta^1\right)^v}{z_1 x_i^{\frac{v}{1-v}} \left(\zeta^i\right)^v} \left(\frac{z_1 x_1 \left(\zeta^1\right)^{1-v}}{z_i x_i \left(\zeta^i\right)^{1-v}}\right)^{\frac{\alpha v}{\alpha v-1}} = \left(\frac{z_i}{z_1}\right)^{-\frac{1}{\alpha v-1}} \left(\frac{x_i}{x_1}\right)^{-\frac{v}{1-v}-\frac{\alpha v}{\alpha v-1}}$$

$$= \left(\frac{z_i}{z_1}\right)^{-\frac{1}{\alpha v-1}} \left(\frac{x_i}{x_1}\right)^{\frac{v(1-\alpha)}{(\alpha v-1)(1-v)}}$$

(this is a transformation of eq 3.7)

5 Share of Capital Controlled by Foreign Firms

In an interior equilbrium, the share of capital controlled by domestic firms in country i is

$$1-s^i = \frac{L_i {x^i}^{\frac{1}{1-v}} \zeta^i}{L_i {x^i}^{\frac{1}{1-v}} \zeta^i + L_1 m^i {x^1}^{\frac{1}{1-v}} \zeta^1} = \frac{{x^i}^{\frac{1}{1-v}} \zeta^i}{\kappa}$$

where

$$\kappa = x^{i\frac{1}{1-v}}\zeta^i + \frac{L_1}{L_i}m^i x^{1\frac{1}{1-v}}\zeta^1$$

We need to solve for κ using the fact that firms in country 1 must be indifferent about where they locate:

$$(1-\tau_D^1)\pi^1 = (1-\tau_F^i)\pi_F^i$$

$$(1-\tau_D^1)\pi_1 = (1-\tau_F^i)(1-v)z_i x_1^{\frac{1}{1-v}}\kappa^{-v}\left(\frac{K_i}{L_i}\right)^{va}F(e)^{v(1-\alpha)}$$

so

$$\frac{(1-\tau_D^1)\pi_1}{(1-\tau_F^i)(1-v)z_i x_i^{\frac{1}{1-v}}\left(\frac{K_i}{L_i}\right)^{va}F(e)^{v(1-\alpha)}} = \kappa^{-v}$$

noting that (from 3.1)

$$\frac{Y_i}{L_i} = z^i\kappa^{1-v}\left(\frac{K_i}{L_i}\right)^{va}F(e)^{v(1-\alpha)}$$

the previous expression can be rewritten as

$$\frac{(1-\tau_D^1)\pi_1}{(1-\tau_F^i)(1-v)x_i^{\frac{1}{1-v}}\frac{Y_i}{L_i}} = \kappa^{-1}$$

$$1-s^i = \frac{x^{i\frac{1}{1-v}}\zeta^i}{\kappa} = x^{i\frac{1}{1-v}}\zeta^i\frac{(1-\tau_D^1)\pi_1}{(1-\tau_F^i)(1-v)x_1^{\frac{1}{1-v}}\frac{Y_i}{L_i}}$$

and the share controlled by foreign firms is s^i

6 Decomposing TFP

Finally, to get a measure of TFP:

$$\frac{Y_i}{L_i} = z_i F(\overline{e}^i)^{v(1-\alpha)}\kappa^{1-v}\left(\frac{K_i}{L_i}\right)^{va} = z_i F(\overline{e}^i)^{v(1-\alpha)}\left(\frac{x^{i\frac{1}{1-v}}\zeta^i}{x^{i\frac{1}{1-v}}\zeta^i\frac{(1-\tau_D^1)\pi_1}{(1-\tau_F^i)(1-v)x_1^{\frac{1}{1-v}}\frac{Y_i}{L_i}}}\right)^{1-v}\left(\frac{K_i}{L_i}\right)^{va}$$

$$\left(\frac{Y_i}{L_i}\right)^v = z_i F(\overline{e}^i)^{v(1-\alpha)}\left(\frac{(1-\tau_F^i)(1-v)x_1^{\frac{1}{1-v}}}{(1-\tau_D^1)\pi_1}\right)^{1-v}\left(\frac{K_i}{L_i}\right)^{va}$$

$$\left(\frac{Y_i}{L_i}\right) = (z_i)^{\frac{1}{v}}F(\overline{e}^i)^{(1-\alpha)}\left(\frac{(1-\tau_F^i)(1-v)x_1^{\frac{1}{1-v}}}{(1-\tau_D^1)\pi_1}\right)^{\frac{1-v}{v}}\left(\frac{K_i}{L_i}\right)^{a}$$

(this is eq 3.9)

Now we can get an expression for x^i in terms of $1-s^i$.

The two key equations we will use are

$$\frac{Y_i}{L_i} = (z_i)^{\frac{1}{v}} F(\overline{e}^i)^{(1-\alpha)} \left(\frac{(1-\tau_F^i)(1-v)x_1^{\frac{1}{1-v}}}{(1-\tau_D^1)\pi_1} \right)^{\frac{1-v}{v}} \left(\frac{K_i}{L_i}\right)^a$$

and

$$\frac{Y_i}{L_i} = z_i F(\overline{e}^i)^{v(1-\alpha)} \left(L_i x^{i\frac{1}{1-v}} \zeta^i + L_1 m^i x^{1\frac{1}{1-v}} \zeta^1 \right)^{1-v} K_i^{\alpha v} L_i^{v(1-\alpha)-1}$$

The first equation here gives us z_i :

$$z_i = \left(\frac{Y_i}{L_i}\right)^v F(\overline{e}^i)^{-(1-\alpha)v} \left(\frac{(1-\tau_F^i)(1-v)x_1^{\frac{1}{1-v}}}{(1-\tau_D^1)\pi_1} \right)^{v-1} \left(\frac{K_i}{L_i}\right)^{-av}$$

(this is eq. 4.7)

The second equation can be written in terms of $1-s^i$, the share of the economy controlled by loca firms:

$$1-s^i = \frac{L_i x^{i\frac{1}{1-v}} \zeta^i}{L_i x^{i\frac{1}{1-v}} \zeta^i + L_1 m^i x^{1\frac{1}{1-v}} \zeta^1}$$

$$\begin{aligned}
\frac{Y_i}{L_i} &= z_i F(\overline{e}^i)^{v(1-\alpha)} \left(L_i x^{i\frac{1}{1-v}} \zeta^i + L_1 m^i x^{1\frac{1}{1-v}} \zeta^1 \right)^{1-v} K_i^{\alpha v} L_i^{v(1-\alpha)-1} \\
&= z_i F(\overline{e}^i)^{v(1-\alpha)} \left(\frac{L_i x^{i\frac{1}{1-v}} \zeta^i}{(1-s^i)} \right)^{1-v} K_i^{\alpha v} L_i^{v(1-\alpha)-1} \\
&= z_i F(\overline{e}^i)^{v(1-\alpha)} x_i \left(\zeta^i\right)^{1-v} \left(1-s^i\right)^{v-1} K_i^{\alpha v} L_i^{-\alpha v}
\end{aligned}$$

Plugging in the experssion for z_i from above:

$$\begin{aligned}
\frac{Y_i}{L_i} &= \left(\frac{Y_i}{L_i}\right)^v F(\overline{e}^i)^{-(1-\alpha)v} \left(\frac{(1-\tau_F^i)(1-v)x_1^{\frac{1}{1-v}}}{(1-\tau_D^1)\pi_1} \right)^{v-1} \left(\frac{K_i}{L_i}\right)^{-av} F(\overline{e}^i)^{v(1-\alpha)} x_i \left(\zeta^i\right)^{1-v} \left(1-s^i\right)^{v-1} K_i^{\alpha v} L_i^{-\alpha v} \\
x_i &= \left(\frac{Y_i}{L_i}\right)^{1-v} \left(\frac{(1-\tau_F^i)(1-v)}{(1-\tau_D^1)\pi_1} \right)^{1-v} x_1 \left(\zeta^i\right)^{v-1} \left(1-s^i\right)^{1-v}
\end{aligned}$$

(this is eq. 4.5)

7 Identifying x_i and z_i from Data

So now we are pretty much at the point where given data on Y_i, L_i K_i s_i τ_F^i, τ_D^i we can compute country specific values for z_i and x_i.

To do that we first have to figure out the critical values $\overline{e}^i$ and $\overline{e}^1$ so that we can compute $F(\overline{e}^i)$ and ζ^i and ζ^1 and we also need to figure out π_1.

The authors show us how to do this in Section 4. Repeating what they write there:

1. Guess an initial m^1. Normalize $x_1 = 1$
2. Given m^1 compute $\overline{e}^1$ by using the fact that the agent in 1 with $e = \overline{e}^1$ is indifferent between becoming a worker or an entrepreneur (see the expression for w_1 and π_1 above)
3. Given the expression for aggregate output in country 1 compute z_1
4. Compute the occupation thresholds in all other countries by equating returns to working versus starting a business, using the expression for $1 - s^i$ to substitute out the component $L_1 m^i {x^1}^{\frac{1}{1-v}} \zeta^1$.
5. Solve for x_i using eq 4.5.
6. Given x_i and s_i solve for m^i. This can be done using

$$s^i = \frac{m^i L_1 x_1^{\frac{1}{1-v}} \zeta^1}{x_i^{\frac{1}{1-v}} L_i \zeta^i + m^i L_1 x_1^{\frac{1}{1-v}} \zeta^1}$$

7. Solve for z_i using eq 4.7
8. Check whether $\sum_i m^i = 1$ - if not adjust guess for m^1
9. Iterate until convergence