Notes on Kehoe Perri, Econometrica 2002

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There is nothing in these notes that is not in Kehoe Perri NBER Working Paper 7820 or Kehoe and Perri Econometrica 2002. However, I have gathered all the equations together, and added a few more steps in the computations in places.

1 Model

Planner's objective

$$\max \left[\lambda_1 \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) U\left(c_1(s^t), l_1(s^t)\right) + \lambda_2 \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) U\left(c_2(s^t), l_2(s^t)\right) \right]$$

Resource constraints

$$\sum_{i=1,2} \left[c_i(s^t) + k_i(s^t)\right] = \sum_{i=1,2} \left[F(k_i(s^{t-1}), A_i(s^t) l_i(s^t)) + (1-\delta) k_i(s^{t-1})\right] \qquad \forall t, s^t$$

Enforcement constraints

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi(s^r | s^t) U\left(c_i(s^r), l_i(s^r)\right) \geq V_i(k_i(s^{t-1}), s^t) \qquad \forall i, t, s^t$$

Autarky problem

$$V_i(k_i(s^{t-1}), s^t) = \max \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi(s^r | s^t) U(c_i(s^r), l_i(s^r))$$

subject to

$$c_i(s^r) + k_i(s^r) \leq F(k_i(s^{r-1}), A_i(s^r) l_i(s^r)) + (1-\delta) k_i(s^{r-1}) \qquad \forall r, s^r$$

Lagrangian for planner's problem

$$\max_{\{c_i(s^t),l_i(s^t),k_i(s^t)\}} \lambda_1 \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t)U\left(c_1(s^t),l_1(s^t)\right)+\lambda_2\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t)U\left(c_2(s^t),l_2(s^t)\right)$$
$$+\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t)\sum_{i=1,2}\mu_i(s^t)\left[\sum_{r=t}^{\infty}\sum_{s^r}\beta^{r-t}\pi(s^r|s^t)U\left(c_i(s^r),l_i(s^r)\right)-V_i(k_i(s^{t-1}),s^t)\right]$$
$$+\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t)\gamma(s^t)\left[\sum_{i=1,2}\left[F(k_i(s^{t-1}),A_i(s^t)l_i(s^t))+(1-\delta)k_i(s^{t-1})\right]-\sum_{i=1,2}\left[c_i(s^t)+k_i(s^t)\right]\right]$$

Now note that $\pi(s^r)=\pi(s^r|s^t)\pi(s^t)$.

Note also that there is a 'partial summation formula of Abel' which points out that

$$\sum_{t=0}^{\infty}\beta^t\mu_t\sum_{r=t}^{\infty}\beta^{r-t}u(c_r)=\sum_{t=0}^{\infty}\beta^tM_tu(c_t)$$

where

$$M_t=M_{t-1}+\mu_t,\ M_{-1}=0$$

Lets apply this to the Lagrangian

$$\max_{\{c_i(s^t),l_i(s^t),k_i(s^t)\}}+\sum_{t=0}^{\infty}\sum_{s^t}\sum_{i=1,2}\beta^t\pi(s^t)\left[M_i(s^t)U\left(c_i(s^t),l_i(s^t)\right)-\mu_i(s^t)V_i(k_i(s^{t-1}),s^t)\right]$$
$$+\text{ resource constraints}$$

where

$$M_i(s^t)=M_i(s^{t-1})+\mu_i(s^t)$$
$$M_i(s^{-1})=\lambda_i$$

KP instead choose to write

$$\max_{\{c_i(s^t),l_i(s^t),k_i(s^t)\}}\sum_{t=0}^{\infty}\sum_{s^t}\sum_{i=1,2}\beta^t\pi(s^t)\left[M_i(s^{t-1})U\left(c_i(s^t),l_i(s^t)\right)+\mu_i(s^t)\left[U\left(c_i(s^t),l_i(s^t)\right)-V_i(k_i(s^{t-1}),s^t\right.\right.$$
$$+\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t)\gamma(s^t)\left[\sum_{i=1,2}\left[F(k_i(s^{t-1}),A_i(s^t)l_i(s^t))+(1-\delta)k_i(s^{t-1})\right]-\sum_{i=1,2}\left[c_i(s^t)+k_i(s^t)\right]\right]$$

(we will use this to compute first order conditions)

2 First order conditions

1. $c_i(s^t)$

$$\beta^t\pi(s^t)M_i(s^{t-1})U_{ic}(s^t)+\beta^t\pi(s^t)\mu_i(s^t)U_{ic}(s^t)-\beta^t\pi(s^t)\gamma(s^t)=0$$

Combining the FOCs for the two countries, we get

$$U_{1c}(s^t)\left(M_1(s^{t-1})+\mu_1(s^t)\right)=U_{2c}(s^t)\left(M_2(s^{t-1})+\mu_2(s^t)\right)$$

or

$$\frac{U_{1c}(s^t)}{U_{2c}(s^t)}=\frac{M_2(s^{t-1})+\mu_2(s^t)}{M_1(s^{t-1})+\mu_1(s^t)}$$

(note this implies ratio of sum of past multipliers defines relative marginal values - implications without participation constraints)

2. $l_i(s^t)$

$$\beta^t\pi(s^t)M_i(s^t)U_{il}(s^t)+\beta^t\pi(s^t)\mu_i(s^t)U_{il}(s^t)+\beta^t\pi(s^t)\gamma(s^t)F_{il}(s^t)=0$$

Combining this with the FOC for $c_i(s^t)$ gives

$$\begin{aligned}-\frac{U_{il}(s^t)\left(M_i(s^t)+\mu_i(s^t)\right)}{F_{il}(s^t)} &= U_{ic}(s^t)\left(M_i(s^t)+\mu_i(s^t)\right)\\ or\ -\frac{U_{il}(s^t)}{F_{il}(s^t)} &= U_{ic}(s^t)\end{aligned}$$

3. $k_i(s^t)$

$$-\beta^t\pi(s^t)\gamma(s^t)+\beta^{t+1}\sum_{s_{t+1}}\pi(s^t,s_{t+1})\left[\gamma(s^t,s_{t+1})r_i(s^t,s_{t+1})-\mu_i(s^t,s_{t+1})V_{ik}(k_i(s^t),\left(s^t,s_{t+1}\right))\right]=0$$

$$\begin{aligned}&U_{ic}(s^t)\left(M_i(s^{t-1})+\mu_i(s^t)\right)\\ =\ &\beta\sum_{s_{t+1}}\frac{\pi(s^t,s_{t+1})}{\pi(s^t)}\left[U_{ic}(s^{t+1})\left(M_i(s^t)+\mu_i(s^{t+1})\right)r_i(s^{t+1})-\mu_i(s^t,s_{t+1})V_{ik}(k_i(s^t),\left(s^t,s_{t+1}\right))\right]\end{aligned}$$

$$\begin{aligned}U_{ic}(s^t) &= \beta\sum_{s_{t+1}}\pi(s_{t+1}|s^t)\left[U_{ic}(s^{t+1})\frac{M_i(s^{t+1})}{M_i(s^t)}r_i(s^t,s_{t+1})-\frac{\mu_i(s^t,s_{t+1})}{M_i(s^t)}V_{ik}(k_i(s^t),(s^t,s_{t+1}))\right]\\ &= \beta\sum_{s_{t+1}}\pi(s_{t+1}|s^t)\left[U_{ic}(s^{t+1})r_i(s^{t+1})+\frac{\mu_i(s^{t+1})}{M_i(s^t)}\left[U_{ic}(s^{t+1})r_i(s^{t+1})-V_{ik}(k_i(s^t),(s^t,s_{t+1}))\right]\right]\end{aligned}$$

where

$$r_i(s^{t+1})=F_{ik}(s^{t+1})+(1-\delta)$$

From the autarky problem,

$$V_{ik}(k_i(s^t),s^{t+1})=U_{ic}^{Aut}(s^{t+1})r_i^{Aut}(s^{t+1})$$

Comment: If no constraints are binding, the capital accumulation FOC is standard. Otherwise things look different. If you think agent 1's constraint might bind tomorrow ($\mu_1(s^{t+1}) > 0$) then the reason must be that you think autarky consumption will be high, so $V_{1k}(k_1(s^t), s^{t+1})$ will be low - lower than $U_{ic}(s^{t+1})r_i(s^{t+1})$ - so the RHS of the inter-temporal FOC is larger that in the absence of the enforcement constraint. One might think that this would suggest the planner should increase $k_1(s^t)$, to reduce $r_i(s^{t+1})$. However, increasing $k_1(s^t)$ drives $V_{1k}(k_1(s^t), s^{t+1})$ further down, making the RHS even larger. Thus the planner actually reduces $k_1(s^t)$. This is (roughly) the intuition for why in this economy, net exports are pro-cyclial: when country 1 has a positive persistent productivity shock - and is therefore tempted to default - the planner reduces $k_1(s^t)$ and increases $k_2(s^t)$. The reduction is $k_1(s^t)$ reduces the value of default in the next period, and the increased $k_2(s^t)$ makes potential future transfers from country 2 larger, increasing value inside the contract.

3 New State Variables

As in the simpler Kocherlakota economy, Kehoe and Perri then define some new variables

$$v_i(s^t) = \frac{\mu_i(s^t)}{M_i(s^t)} \Rightarrow 1 - v_i(s^t) = \frac{M_i(s^t) - \mu_i(s^t)}{M_i(s^t)} = \frac{M_i(s^{t-1})}{M_i(s^t)}$$

$$z(s^t) = \frac{M_2(s^t)}{M_1(s^t)}$$

Then

$$M_i(s^t) = \frac{M_i(s^{t-1})}{1 - v_i(s^t)}$$

and thus

$$z(s^t) = \frac{1 - v_1(s^t)}{1 - v_2(s^t)} z(s^{t-1})$$

Now the first order conditions for consumption and capital can be rewritten as

$$\frac{U_{1c}(s^t)}{U_{2c}(s^t)} = \frac{M_2(s^t) + \mu_2(s^t)}{M_1(s^{t-1}) + \mu_1(s^t)} = z(s^t) = \frac{1 - v_1(s^t)}{1 - v_2(s^t)} z(s^{t-1})$$

and

$$U_{ic}(s^t) = \beta \sum_{s_{t+1}} \pi(s_{t+1}|s^t) \left[\frac{U_{ic}(s^{t+1}) r_i(s^{t+1})}{1 - v_i(s^{t+1})} - \frac{v_i(s^{t+1})}{1 - v_i(s^{t+1})} V_{ik}(k_i(s^t), s^{t+1}) \right]$$

Kehoe and Perri focus on Markov shocks, so that $\pi(s^t|s^{t-1}) = \pi(s_t|s_{t-1})$.

Then look for decision rules $c_i(x_t)$, $l_i(x_t)$, $k_i(x_t)$ together with policy rules for $z(x_t)$ and $v_i(x_t)$ where the state $x_t = \left(z(s^{t-1}), k_1(s^{t-1}), k_2(s^{t-1}), s_t\right)$. Thus this is the same state space we used to solve the Kocherlakota model, except that $k_1(s^{t-1})$ and $k_2(s^{t-1})$ have been added.

4 Computation procedure

Let $x = (z, k_1, k_2, s)$ be the state (note that the notation is a bit confusing, since the understanding is that z, k_1 and k_2 are inherited from the previous period, and do not depend on the current state s).

Define also value functions

$$W_i(x) = U(c_i(x), l_i(x)) + \beta \sum_{s'} \pi(s'|s) W_i(x')$$

Define a grid X on the state space (i.e a three dimensional grid over the three continuous variables z, k_1 and k_2 for each possible (discrete) value for s.

Guess $\left\{c_i^0(x), l_i^0(x), k_i'^0(x), z'^0(x), v_i^0(x), W_i^0(x)\right\}$ $\forall x \in X$. One good initial guess is the solution to the planning problem without enforcement constraints. Note that these allocations correspond to the allocations that would emerge in a decentralized economy with a complete set of asset markets and no enforcement problems.

Now procede to update the guess. Suppose, we are on the n^{th} iteration in updating the vector of unknown functions, and suppose we are at point q in the grid.

First assume neither enforcement constraint binds. Thus immediately $v_i(q) = 0$ and $z'(q) = z$.

Compute $(c_i(q), l_i(q), k_i'(q))$ (6 numbers) that satisfy

1. The consumption leisure FOC for both agents (2 equations)

$$\frac{U_{il}(q)}{F_{il}(q)} = U_{ic}(q)$$

2. The expression for z (1 equation)

$$\frac{U_{1c}(q)}{U_{2c}(q)} = z'(q)$$

3. The world resource constraint (1 equation)

$$\sum_{i=1,2} [c_i(q) + k_i'(q)] = \sum_{i=1,2} [F(k_i(q), A_i(q) l_i(q)) + (1-\delta) k_i(q)]$$

4. The FOCs for capital (2 equations)

$$\begin{aligned} & U_c(c_i(q), l_i(q)) \\ = \quad & \beta \sum_{s\prime} \pi(s'|s) \left[U_c(c_i^{n-1}(x'), l_i^{n-1}(x')) \frac{F_k(k_i'(q), A(s') l_i^{n-1}(x')) + 1 - \delta}{1 - v_i^{n-1}(x')} - \frac{v_i^{n-1}(x')}{1 - v_i^{n-1}(x')} V_{ik}(k'(q), s') \right] \end{aligned}$$

   Note that, to evaluate c_i, l_i and v_i in the next period, we use the old guesses for the decision rules policy functions (superscript $n-1$). The 'unknowns' in this equation are $c_i(q)$, $l_i(q)$ and $k_i'(q)$. Note, however, that although we use the old functions for next period variables, we do evaluate them at the correct point: $x' = (z'(q), k_1'(q), k_2'(q), s')$.

Thus at this point we solve a system of 6 non-linear equations in 6 unknowns.

Then we check whether either of the two enforcement constraints is violated. If neither is, we are done with this grid point.

Alternatively suppose one of the enforcement constraints is violated (say the one for country 1). Then we solve a different system of equations. Now we look for $(c_i(q), l_i(q), k_i'(q), v_1(q), z'(q))$ (8 numbers) that satisfy

1. The consumption leisure FOC for both agents (2 equations)

$$\frac{U_{il}(q)}{F_{il}(q)} = U_{ic}(q)$$

2. The expression for z (1 equation)

$$\frac{U_{1c}(q)}{U_{2c}(q)} = z'(q)$$

3. The law of motion for z (1 equation)

$$z\prime(q) = \frac{1 - v_1(q)}{1} z$$

4. The world resource constraint (1 equation)

$$\sum_{i=1,2} [c_i(q) + k_i'(q)] = \sum_{i=1,2} [F(k_i(q), A_i(q)l_i(q)) + (1-\delta)k_i(q)]$$

5. The FOCs for capital (2 equations)

$$U_c(c_i(q), l_i(q))$$
$$= \beta \sum_{s\prime} \pi(s\prime|s) \left[U_c(c_i^{n-1}(x'), l_i^{n-1}(x')) \frac{F_k(k_i'(q), A(s')l_i^{n-1}(x')) + 1 - \delta}{1 - v_i^{n-1}(x')} - \frac{v_i^{n-1}(x')}{1 - v_i^{n-1}(x')} V_{ik}(k'(q), s') \right]$$

6. The enforcment constraint for country 1 is an equality (1 equation)

$$U_c(c_i(q), l_i(q)) + \beta \sum_{s'} \pi(s'|s) W_i^{n-1}(x') = V_i(k_i, s)$$

Note that in equilibrium, if the planner is doing its job, it will not be the case the both enforcement constraints bind simultaneously.

Once decision rules have been updated at one point in the grid, we move to the next point, and continue until decision rules have been updated. Now (and only now) we can update the value functions, for each $x \in X$. For example, for $x = q$

$$W_i^n(q) = U(c_i^n(q), l_i^n(q)) + \beta \sum_{s'} \pi(s'|s) W_i^{n-1}(x'(q))$$

If the new value functions are cloes to the old for every x, we are done. Otherwise, we need to repeat the entire process.

It is important the the initial guess for the value functions is uniformly greater than or equal to the true value of the value function.