Two Useful Computational Tricks for Solving Consumption-Savings Problems (rough notes)

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No trade result in HSV QJE (and Constantinides and Duffie) works because of special assumptions

Easy to break:

- Change nature of shocks
- Introduce deterministic life-cycle profile for wages and / or retirement
- Make risk vary with age

We can always simply assume no borrowing / lending

But that is extreme and unrealistic

How to incorporate savings?

Must turn numerical

But can compute things in a pretty simple way.

Consider life-cycle model

- Can incorporate life-cycle wage profile, retirement etc

For simplicity, rule out other model elements (exogenous labor supply etc)

People live from age $j = 0$ to age $j = J$

Assume shocks to earnings are unit root

$$z_{j+1} = z_j + \omega_{j+1}$$

$$\omega_{j+1} \tilde{} N\left(-\frac{v_\omega}{2}, v_\omega\right)$$
$$z_0 \tilde{} N\left(-\frac{v_0}{2}, v_0\right)$$

Earnings at age j are

$$y_j = x_j \exp(z_j)$$

where x_j is a deterministic life-cycle profile

Assume borrowing constraint of the form

$$a_{j+1} \geq \kappa_j \exp(z_j)$$

where κ_j is a possibly age-specific exogenous parameter. Assume $\kappa_J = 0$.

Household solves

$$\max_{\{c_j,s_j\}} E\left[\sum_{j=0}^{J}\beta^j u(c_j)\right]$$

s.t.

$$\begin{aligned}
c_j + s_j &= x_j \exp(z_j) + Ra_j \\
s_j &\geq \kappa_j \exp(z_j) \\
a_{j+1} &= s_j \\
a_0 &= 0,\ z_0 \text{ given}
\end{aligned}$$

Assume utility function is homothetic, e.g.,

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

FOC is

$$\begin{aligned}
u'(c_j) &\geq \beta RE\left[u'(c_{j+1})\right] \\
&= \text{ if borrowing constraint does not bind}
\end{aligned}$$

1 Trick number one

(requires unit root shocks, homothetic preferences, and borrowing constraints proportional to income)

Consider two individuals, one with productivity $\exp(z_j)$ and wealth a_j and another with productivity $\lambda\exp(z_j)$ and wealth λa_j. If the optimal savings choice for the first agent is $a^*(z_j, a_j)$, the optimal savings choice for the second is $\lambda a^*(z_j, a_j)$. Put differently, the only relevant state variable for the choice for $\frac{a_{j+1}}{z_j}$ is the ratio $\frac{a_j}{z_j}$. Reducing the state space this way is very useful.

Let's take a transformation of variables of this form to the equations defining a solution to the agent's problem, using tilda's to denote that a variable has been divided by $\exp(z_j)$.

The transformed equations describing a solution to the household problem are:

$$\begin{aligned}
\tilde{c}_j + \tilde{s}_j &= x_j + R\tilde{a}_j \\
\tilde{s}_j &\geq \kappa_j
\end{aligned}$$

where

$$\tilde{a}_{j+1} = \frac{a_{j+1}}{\exp(z_{j+1})} = \tilde{s}_j \exp(-\omega_{j+1})$$

FOC is

$$\begin{aligned}
u'(\tilde{c}_j) &\geq \beta RE\left[u'\left(\tilde{c}_{j+1}\exp(\omega_{j+1})\right)\right] \\
&= \text{ if constraint does not bind}
\end{aligned}$$

Nice thing is that we have dropped z_j as a state, and only have $\tilde{a}_j$.

2 Trick number two

Standard way to solve such models: start at last period and work backwards. Will always do this

Also need to discretize ω shock: assume it can take N possible values

Compare 2 ways to solve this problem

2.1 Standard way:

1. Construct grid on $\tilde{a}_j$ at each age $j : \left\{\tilde{a}_j^k\right\}_{k=1}^K$
2. Last period solution is, for each point on grid $\tilde{a}_j^k$,
$$\tilde{s}_J\left(\tilde{a}_{J-1}^k\right) = 0$$
which implies
$$\tilde{c}_J \tilde{a}_J^k = x_J + R\tilde{a}_J^k$$
3. Given grid on ω and R can evaluate RHS of Euler equation at each point on grid
4. Now turn to $j = J - 1$
5. At each grid point $\tilde{a}_{J-1}^k$, search for $\tilde{s}_{J-1}\left(\tilde{a}_{J-1}^k\right)$ that satisfies intertemporal FOC. This is the computationally intensive step:
   (a) Guess an $\tilde{s}_{J-1}\left(\tilde{a}_{J-1}^k\right)$, which implies $\tilde{c}_{J-1}\left(\tilde{a}_{J-1}^k\right)$ and a value for LHS of the EE (quick)
   (b) For each possible ω_J compute $\tilde{a}_J$
   $$\tilde{a}_J = \tilde{s}_{J-1}\left(\tilde{a}_{J-1}^k\right)\exp(-\omega_J)$$
   (c) Find which points in the grid $\left\{\tilde{a}_J^k\right\}$ the value $\tilde{a}_J$ lies in between (slow)
   (d) Interpolate between these grid points to estimate $\tilde{s}_J\left(\tilde{a}_J\right)$ and thence $\tilde{c}_J$ and $u'\left(\tilde{c}_J\exp(\omega_J)\right)$
   (e) Sum over ω_J to figure out RHS of EE
   (f) Check whether Euler is satisfied, if not adjust guess (slow)

A time-consuming procedure.

Also difficult to deal with borrowing constraint. Consumption and savings rule will tend to kink at borrowing constraint, and so therefore will marginal utility function

But linear interpolation won't capture this kink

Will be OK only if by chance we put a grid point at exactly the point where the kink is located

2.2 Alternative endogenous grid point method

Quite similar, except in how grid is constructed.

Think about working with grids on savings, instead of a grid on current wealth

1. Construct grid on s_{J-1}, $\left\{s_{J-1}^k\right\}_{k=1}^K$. We know where to start this grid: at the borrowing constraint κ_{J-1}.

2. For each value for s_{J-1}^k and for each value fo ω_J we can construct

$$\tilde{a}_J = s_{J-1}^k \exp(-\omega_J)$$

$$\tilde{c}_J = x_J + R\tilde{a}_J$$

   and thus, summing over ω_J

$$\beta RE\left[u'\left(\tilde{c}_J \exp(\omega_J)\right)\right]$$

3. Now it is a simple matter for compute (closed-form) the value for $\tilde{c}_{J-1}$ that satisfies the FOC with equality

$$u'(\tilde{c}_{J-1}) = \beta RE\left[u'\left(\tilde{c}_J \exp(\omega_J)\right)\right]$$

   e.g., if utility is logarithmic

$$\tilde{c}_{J-1} = \frac{1}{\beta RE\left[\frac{1}{\tilde{c}_J \exp(\omega_J)}\right]}$$

4. Now we can ask: how much wealth must the agent have at $J-1$ to be able to afford $\tilde{c}_{J-1}$ and s_{J-1}^k? From the budget constraint

$$\tilde{c}_{J-1} + s_{J-1} = x_{J-1} + R\tilde{a}_{J-1}$$

   so we can immediately compute $\tilde{a}_{J-1}$ in closed form.

5. So given a grid on (exogenous values for) s_{J-1} we have constructed a vector of corresponding endogenous values for $\tilde{a}_{J-1}$. Note that the lowest value for savings in our original grid on savings at $J-1$, κ_{J-1}, delivers a particular value for $\tilde{a}_{J-1}$ which we can denote $\tilde{a}_{J-1}^*$.

6. Moving forward it will be convenient to have (endogenous) values for s_{J-1} on an exogenous grid for $\tilde{a}_{J-1}$. We therefore need to reconstruct the savings rule on a grid for $\tilde{a}_{J-1}$.

   (a) This new grid should start at the lowest possible value for $\tilde{a}_{J-1}$ which corresponds to $\kappa_{J-2}\exp(-\omega_{\max})$.

(b) We know that the savings rule at $J-1$ has the property that $s(\tilde{a}_{J-1}) = \kappa_{J-1}$ for all $\tilde{a}_{J-1} \leq \tilde{a}^*_{J-1}$. When we construct the new grid on $\tilde{a}_{J-1}$ (over which the savings rule is approximated) we should place a grid point at $\tilde{a}^*_{J-1}$ to capture this kink.

(c) To approximate optimal savings choices at other grid points we will need to do some interpolation.

7. We can now move toward solving for savings choices at $J-2$.

   (a) Again, start by constructing a grid on savings at $J-2$, $\left\{s^k_{J-1}\right\}_{k=1}^K$

   (b) For each point on this grid we construct, for each ω_{J-1}

$$\tilde{a}_{J-1} = s^k_{J-2} \exp(-\omega_{J-1})$$

$$\tilde{c}_{J-1} = x_{J-1} + R\tilde{a}_{J-1} - s_{J-1}(\tilde{a}_{J-1})$$

   where here we use the decision rule from the previous step

   (c) Integrating over ω we get $\beta RE\left[u'\left(\tilde{c}_{J-1}\exp(\omega_{J-1})\right)\right]$

   (d) Again we can use the intertemporal FOC (at equality) to compute $\tilde{c}_{J-2}$ and then $\tilde{a}_{J-2}$