1 Hall and Sargent Versions of Barro (79) and Lucas and Stokey (83)

Let's collect the Hall and Sargent model equations

There is a quadratic loss function

$$E\sum_{t=0}^{\infty}\beta^t T_t^2$$

There is a stochastic process for G_t that is a Markov process and a function of an underlying state z_t : $G_t = G(z_t)$

$$PVG_t = E\left[\sum_{j=0}^{\infty}\beta^t G_{t+j}\right]$$

In the Barro version of the debt market

$$T_t + a_t = \frac{1}{R}a_{t+1} + G_t$$

where $R = \beta^{-1}$ is constant real rate and a_t is government assets

In the Lucas and Stokey version of the debt market

$$T_t + a_{t-1}(z_t) = G_t + \int q_t(z_{t+1}|z_t)a_t(z_{t+1}|z_t)dz_{t+1}$$

These state contingent bond prices are actuarially fair

$$q_t(z_{t+1}|z_t) = \beta\phi(z_{t+1}|z_t)$$

so the budget constraint simplifies to

$$T_t + a_{t-1}(z_t) = G_t + \beta E\left[a_{t+1}|z_t\right]$$

Ex post return to portfolio is

$$R_t(z_{t+1}|z_t) = \frac{a_t(z_{t+1}|z_t)}{\beta E\left[a_{t+1}|z_t\right]}$$

So the expected return on the portfolio is β^{-1}

The optimal policy in the Barro model is the same as in the consumption savings model with quadratic utility:

$$T_t = (1-\beta)\left[PVG_t - a_t\right]$$

which will have the property that taxes follow a random walk, i.e.,

$$T_t = E_t\left[T_{t+1}\right]$$

We also know that

$$\begin{aligned}\Delta T_t &= T_t - T_{t-1} = T_t - E_{t-1}[T_t] \\ &= (1-\beta)\left[PVG_t - E_{t-1}\left[PVG_t\right]\right]\end{aligned}$$

so changes in taxes are driven by innovations to the present value of G_t

Suppose

$$\begin{aligned}G_t &= g_{1t} + g_{2t} \\ g_{1,t+1} &= g_{1t} + \sigma_1\omega_{1,t+1} \\ g_{2,t+1} &= \sigma_2\omega_{2,t+1}\end{aligned}$$

Then

$$\begin{aligned}PVG_t &= \frac{g_{1t}}{1-\beta} + g_{2t} \\ E_{t-1}\left[PVG_t\right] &= \frac{g_{1,t-1}}{1-\beta}\end{aligned}$$

$$\Delta T_t = \sigma_1\omega_{1,t+1} + (1-\beta)\sigma_2\omega_{2,t+1}$$

From the budget constraints

$$\begin{aligned}a_{t+1} &= RT_t + Ra_t - RG_t \\ a_{t+1} - a_t &= R\left(g_{1t} + (1-\beta)g_{2t} - (1-\beta)a_t\right) + (R-1)a_t - R\left(g_{1t} + g_{2t}\right) \\ &= -R\beta g_{2t} - R(1-\beta)a_t + (R-1)a_t \\ &= -\sigma_2\omega_{2,t+1}\end{aligned}$$

In the alternative Lucas-Stokey model, we have

$$T_t = T_0 = (1-\beta)(PVG_0 - a_0)$$

And at each date date and state z_t the asset purchase $a_{t-1}(z_t)$ satisfies

$$\begin{aligned}T_0 &= (1-\beta)(PVG_t - a_{t-1}(z_t)) \\ a_{t-1}(z_t) &= PVG_t - \frac{T_0}{1-\beta} \\ &= \frac{g_{1t}}{1-\beta} + g_{2t} - \frac{T_0}{1-\beta}\end{aligned}$$

2 Hall and Sargent's Historical Accounting

Budget constraint

$$B_t = B_{t-1} + r_{t-1,t}B_{t-1} + G_t - T_t - (M_t - M_{t-1})$$

where everything is nominal.

B_{t-1} is the nominal value of all debt at date $t-1$ (not the book or face value)

In reality there are bonds of lots of different maturities, and each offer different nominal returns between $t-1$ and t.

So $r_{t-1,t}$ is the single interest rate that delivers the observed return on the total portfolio

We can divide each piece by nominal GDP

$$\frac{B_t}{Y_t} = \frac{B_{t-1}}{Y_{t-1}}\frac{Y_{t-1}}{Y_t} + r_{t-1,t}\frac{B_{t-1}}{Y_{t-1}}\frac{Y_{t-1}}{Y_t} + \frac{G_t}{Y_t} - \frac{T_t}{Y_t} - \frac{M_t - M_{t-1}}{Y_t}$$

Now

$$\frac{Y_{t-1}}{Y_t} = \frac{1}{(1+g_{t-1,t})(1+\pi_{t-1,t})} \simeq 1 - g_{t-1,t} - \pi_{t-1,t}$$

so we can write

$$\begin{aligned}
\frac{B_t}{Y_t} &= \frac{B_{t-1}}{Y_{t-1}} + (g_{t-1,t}+\pi_{t-1,t})\frac{B_{t-1}}{Y_{t-1}} + r_{t-1,t}\frac{B_{t-1}}{Y_{t-1}} + r_{t-1,t}\frac{B_{t-1}}{Y_{t-1}}(g_{t-1,t}+\pi_{t-1,t}) \\
&\quad + \frac{G_t}{Y_t} - \frac{T_t}{Y_t} - \frac{M_t - M_{t-1}}{Y_t} \\
\frac{G_t}{Y_t} + r_{t-1,t}\frac{B_{t-1}}{Y_{t-1}} &= \frac{T_t}{Y_t} + \left(\frac{B_t}{Y_t} - \frac{B_{t-1}}{Y_{t-1}}\right) + \left(\frac{M_t - M_{t-1}}{Y_t}\right) \\
&\quad + g_{t-1,t}\frac{B_{t-1}}{Y_{t-1}} + \pi_{t-1,t}\frac{B_{t-1}}{Y_{t-1}} + r_{t-1,t}\frac{B_{t-1}}{Y_{t-1}}(g_{t-1,t}+\pi_{t-1,t})
\end{aligned}$$

The LHS is expenditure that must be financed

Sources of financing are: (1) taxes net of transfers, (2) growth in debt to GDP ratio, (3) new money, (3) debt-to-output reduction through economic growth, (4) debt to output reduction through inflation, (5) a cross term (will be small)