Notes on Mirrlees Taxation (rough)

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1 Problem Setup

Static problem:
Agents differ by productivity θ
I values for productivity $\theta_1, ..., \theta_I$
Fraction π_i of each type
Preferences

$$\begin{aligned} U_i &= u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \\ & \log(c_i) - \frac{\left(\frac{y_i}{\theta_i}\right)^{1+\sigma}}{1+\sigma} \end{aligned}$$

Planner must raise revenue to finance G
Planner puts weight W_i on type i s.t. $\sum_i W_i \pi_i = 1$
An allocation is a vector $\{(c_i, y_i)\}_{i=1}^{I}$
Social welfare is given by

$$\sum_i W_i \pi_i \left\{ u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \right\}$$

Planner can observe y, but not θ
So taxes must be a function of y
Planner's problem is therefore to choose a tax function $T(y)$ such that when agents take this schedule as given and solve

$$\max_{\{c_i, y_i\}} \left\{ u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \right\}$$

$$\text{s.t. } c_i = y_i - T(y_i)$$

the resulting allocations maximize social welfare.

Suppose $v(x) = (1+\sigma)^{-1} x^{1+\sigma}$. Model could be interpred as model in which all agents have same wage, but different disutilities of work. Thus, interpretations in which workers have different labor productivity but identical preferences versus interpretation in which they are equally productive but have different disutility of work are formally identical. The idea is that in terms of individual choices it doesn't matter whether it is costly for a worker to deliver output because she must work a lot of hours or because she just hates working. (Though a planner might feel differently about these two people in terms of Pareto weights)

2 Mirrlees' Clever Idea

Now the problem is that the optimal T could be a very complicated non-parametric function. How are we supposed to solve for it?

Mirrlees' clever idea.

Instead of thinking of planner picking T think of planner picking allocations directly.

In particular think of planner as offering a menu of different choices $\{(c_i, y_i)\}$ with one pair in this menu intended for each type. The planner can say:

"If you produce income y_i (which I can observe) then you must pay a tax $y_i - c_i$."

But the planner cannot force agents to choose the pair intended for their type, because type is not observed

Thus the planner must incentivize choosing the appropriate allocation by making sure that each type weakly prefers to pick their intended allocation

Thus the Mirrlees problem is

$$\begin{aligned} & \max_{\{c_i, y_i\}} \sum_i W_i \pi_i \left\{ u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \right\} \\ & s.t. \\ u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \quad \geq \quad & u(c_j) - v\left(\frac{y_j}{\theta_i}\right) \text{ for all } i, j \\ \sum_i \pi_i c_i + G \quad = \quad & \sum_i \pi_i y_i \end{aligned}$$

There are lots of incentive constraints!

Fortunately most of them will not be binding

Suppose planner wants to redistribute downwards => lower taxes on less productive agents => possible incentive to pretend to be less productive, but no incentive to pretend to be more productive => only downward IC constraints will bind

In fact, only local downward constraints will bind.

So we can simplify the problem to

$$\max_{\{c_i,y_i\}} \sum_i W_i \pi_i \left\{ u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \right\}$$

$$\pi_i \mu_i \quad : \quad u(c_i) - v\left(\frac{y_i}{\theta_i}\right) \geq u(c_{i-1}) - v\left(\frac{y_{i-1}}{\theta_i}\right) \text{ for } i = 2, ..., I$$

$$\lambda \quad : \quad \sum_i \pi_i c_i + G \leq \sum_i \pi_i y_i$$

FOCs (recall no IC constraint for $i = 1 => \mu_1 = 0$)

$$c_i \quad : \quad W_i \pi_i u'(c_i) + \pi_i \mu_i u'(c_i) - \lambda \pi_i + \pi_{i+1} \mu_{i+1} u'(c_i) = 0 \text{ for } i = 1, ..., I-1$$

$$y_i \quad : \quad -W_i \pi_i v'\left(\frac{y_i}{\theta_i}\right)\frac{1}{\theta_i} - \pi_i \mu_i v'\left(\frac{y_i}{\theta_i}\right)\frac{1}{\theta_i} + \lambda \pi_i + \pi_{i+1}\mu_{i+1} v'\left(\frac{y_i}{\theta_{i+1}}\right)\frac{1}{\theta_{i+1}} = 0 \text{ for } i = 1, ..., I-1$$

$$c_i \quad : \quad W_i \pi_i u'(c_i) + \pi_i \mu_i u'(c_i) - \lambda \pi_i = 0 \text{ for } i = I$$

$$y_i \quad : \quad -W_i \pi_i v'\left(\frac{y_i}{\theta_i}\right)\frac{1}{\theta_i} - \pi_i \mu_i v'\left(\frac{y_i}{\theta_i}\right)\frac{1}{\theta_i} + \lambda \pi_i = 0 \text{ for } i = I$$

$2I + I - 1 + 1$ unknowns: $\{c_i, y_i\}_{i=1}^I$, $\{\mu_i\}_{i=2}^I$, λ

$2I$ FOCs + I constraints: $I - 1$ IC constraints and the resource constraint

This problem will have a solution. How can we decentralize it? We need to come up with a tax system such that taxes depend on earnings, and all agents are on their FOC

$$u'(c_i)\theta_i(1 - T'(y_i)) = v'\left(\frac{y_i}{\theta_i}\right)$$

and

$$c_i = y_i - T(y_i)$$

Note that marginal and average tax rates are only exactly pinned down at grid points.

FOCs

$$c_i \quad : \quad W_i \pi_i u'(c_i) + \pi_i \mu_i u'(c_i) - \pi_{i+1}\mu_{i+1} u'(c_i) - \lambda \pi_i = 0$$

$$y_i \quad : \quad -W_i \pi_i v'\left(\frac{y_i}{\theta_i}\right)\frac{1}{\theta_i} - \pi_i \mu_i v'\left(\frac{y_i}{\theta_i}\right)\frac{1}{\theta_i} + \pi_{i+1}\mu_{i+1} v'\left(\frac{y_i}{\theta_{i+1}}\right)\frac{1}{\theta_{i+1}} + \lambda \pi_i = 0$$

$$u'(c_i)\theta_i \frac{W_i \pi_i + \pi_i \mu_i - \pi_{i+1}\mu_{i+1}}{W_i \pi_i + \pi_i \mu_i - \pi_{i+1}\mu_{i+1} \frac{v'\left(\frac{y_i}{\theta_{i+1}}\right)\frac{1}{\theta_{i+1}}}{v'\left(\frac{y_i}{\theta_i}\right)\frac{1}{\theta_i}}} = v'\left(\frac{y_i}{\theta_i}\right)$$

3 Zero Marginal Tax at the Top, Positive Elsewhere

Note that for $i = I$ the $i+1$ terms are absent, so can combine the two to give

$$(W_I \pi_I + \pi_I \mu_I)\, \theta_I u'(c_I) = (W_I \pi_I + \pi_I \mu_I)\, v'\left(\frac{y_I}{\theta_I}\right)$$

Compare this to the FOC for the decentralized economy in which individuals face a tax on earnings. It is clear that $T'(y_I) = 0$, so there is no distortion / wedge / implicit tax at the top. A classic result in the literature

More generically, suppose $v(x) = (1+\sigma)^{-1} x^{1+\sigma}$, so $v'(x) = x^{\sigma}$

$$v'\left(\frac{y_i}{\theta_{i+1}}\right) = v'\left(\frac{y_i}{\theta_i}\frac{\theta_i}{\theta_{i+1}}\right) = v'\left(\frac{y_i}{\theta_i}\right)\left(\frac{\theta_i}{\theta_{i+1}}\right)^{\sigma}$$

$$u'(c_i)\theta_i \underbrace{\left(\frac{W_i \pi_i + \pi_i \mu_i - \pi_{i+1}\mu_{i+1}}{W_i \pi_i + \pi_i \mu_i - \pi_{i+1}\mu_{i+1}\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}}\right)}_{wedge = 1 - tax'} = v'\left(\frac{y_i}{\theta_i}\right)$$

Note that wedge is less than one, so tax is positive => labor supply is distorted => can't achieve first best

4 Numerical Solution

How do we solve this numerically?

Guess λ

Guess c_1

Solve for μ_2 for FOC for c_1

Solve for y_1 for FOC for y_1

3 equations (2 FOCs and IC_2) to solve for c_2, y_2 and μ_3

Iterate upwards through the grid

At $I-1$ we solve for μ_I

Then we have 2 FOCs at I to solve for c_I and y_I

Check IC_I and adjust c_1 if not satisfied

Finally check resource constraint and adjust λ

5 Diamond Saez Equation

We have

$$\begin{aligned}
(1-\tau_i) &= \frac{W_i\pi_i + \pi_i\mu_i - \pi_{i+1}\mu_{i+1}}{W_i\pi_i + \pi_i\mu_i - \pi_{i+1}\mu_{i+1}\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}} \\
\tau_i &= \frac{\pi_{i+1}\mu_{i+1} - \pi_{i+1}\mu_{i+1}\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}}{W_i\pi_i + \pi_i\mu_i - \pi_{i+1}\mu_{i+1}\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}} \\
&= \frac{\mu_{i+1}\pi_{i+1}\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right)v'\left(\frac{y_i}{\theta_i}\right)}{\lambda\pi_i\theta_i}
\end{aligned}$$

where the last line subsitutes in the FOC for y_i.

Now let's solve for the multipliers in terms of allocations:

$$\begin{aligned}
W_i\pi_i u'(c_i) + \pi_i\mu_i u'(c_i) - \pi_{i+1}\mu_{i+1}u'(c_i) &= \lambda\pi_i \\
\sum_i W_i\pi_i u'(c_i) + \sum_i \pi_i\mu_i u'(c_i) - \sum_i \pi_{i+1}\mu_{i+1}u'(c_i) &= \lambda \\
\sum_i W_i\pi_i u'(c_i) &= \lambda
\end{aligned}$$

From the same FOC

$$\begin{aligned}
\left(W_i\pi_i + \pi_i\mu_i - \pi_{i+1}\mu_{i+1}\right)u'(c_i) &= \lambda\pi_i \\
\pi_i\mu_i &= \frac{\lambda\pi_i}{u'(c_i)} - W_i\pi_i + \pi_{i+1}\mu_{i+1} \\
\pi_i\mu_i &= \frac{\lambda\pi_i}{u'(c_i)} - W_i\pi_i + \left(\frac{\lambda\pi_{i+1}}{u'(c_{i+1})} - W_{i+1}\pi_{i+1} + \pi_{i+2}\mu_{i+2}\right) \\
&= \lambda\sum_i^I \frac{\pi_i}{u'(c_i)} - \sum_i^I W_i\pi_i
\end{aligned}$$

Plugging the expression for $\pi_i\mu_i$ into the expression for τ_i we have

$$\begin{aligned}
\tau_i &= \frac{\mu_{i+1}\pi_{i+1}\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right)v'\left(\frac{y_i}{\theta_i}\right)}{\lambda\pi_i\theta_i} \\
&= \frac{\left(\lambda\sum_{s=i+1}^I \frac{\pi_s}{u'(c_s)} - \sum_{s=i+1}^I W_s\pi_s\right)\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right)v'\left(\frac{y_i}{\theta_i}\right)}{\lambda\pi_i\theta_i}
\end{aligned}$$

We also know that

$$u'(c_i)\theta_i(1-\tau_i) = v'\left(\frac{y_i}{\theta_i}\right)$$

so

$$\begin{aligned}
\frac{\tau_i}{1-\tau_i} &= \frac{u'(c_i)}{\pi_i}\left(\sum_{s=i+1}^{I}\frac{\pi_s}{u'(c_s)} - \frac{1}{\lambda}\sum_{s=i+1}^{I} W_s\pi_s\right)\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right) \\
&= \frac{u'(c_i)}{\pi_i}\left(\sum_{s=i+1}^{I}\frac{\pi_s}{u'(c_s)} - \frac{1}{\sum_i W_i\pi_i u'(c_i)}\sum_{s=i+1}^{I} W_s\pi_s\right)\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right) \\
&= \left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right)\frac{1}{\pi_i}\left(\sum_{s=i+1}^{I}\pi_s\left(\frac{u'(c_i)}{u'(c_s)} - \frac{u'(c_i)W_s}{\sum_i W_i\pi_i u'(c_i)}\right)\right)
\end{aligned}$$

Now suppose the underlying productivity distribution is truly continuous, with CDF $F(\theta)$ and PDF $f(\theta)$.

Suppose that to construct our discrete approximation we have defined the mass points on the grid as $\pi_i = f(\theta_i)(\theta_{i+1}-\theta_i)$

Let's now construct the continuous productivity version of our discrete Diamond-Saez equation.

The second term is straightforward

$$\sum_{s=i+1}^{I}\pi_s\left(\frac{u'(c_i)}{u'(c_s)} - \frac{u'(c_i)W_s}{\sum_i W_i\pi_i u'(c_i)}\right) \to \int_{s=\theta}^{\infty}\left(\frac{u'(c(\theta))}{u'(c(s))} - \frac{u'(c(\theta))W(s)}{\int W(i)u'(c(i))f(i)di}\right)dF(\theta)$$

What happens to the first term, $\left(1-\left(\frac{\theta_i}{\theta_{i+1}}\right)^{1+\sigma}\right)\frac{1}{f(\theta_i)(\theta_{i+1}-\theta_i)}$ as the grid approaches the continuous limit, i.e., $\frac{\theta_{i+1}}{\theta_i} \to 1$?

By l'Hoptital's rule, it converges to $(1+\sigma)\frac{1}{\theta_i f(\theta_i)}$

Thus, we have an implicit solution for optimal marginal tax rates at each productivity value θ :

$$\frac{\tau(\theta)}{1-\tau(\theta)} = (1+\sigma)\frac{1}{\theta f(\theta)}\int_{s=\theta}^{\infty}\frac{u'(c(\theta))}{u'(c(s))}\left(1-\frac{u'(c(s))W(s)}{\int W(i)u'(c(i))f(i)di}\right)dF(\theta)$$

Look at the second term in parentheses. This captures a distributional incentive to set marginal rates high.

In particular, imagine raising the marginal rate a little at some productivity value θ

$u'(c(s))W(s)$ will be declining in s.

So the entire integral is maximized at $s=\theta$ s.t. $W(\theta)u'(c(\theta)) = \int W(i)u'(c(i))f(i)di$

The equation indicates additional considerations

The larger is σ, the higher will be marginal rates, all else equal (higher σ => labor supply less elastic => taxes less distortionary)

The larger is $\theta f(\theta)$ the lower marginal tax rates will be at θ (high density => lots of agents choices distorted by higher marginal rates).

But there is a limit to how much we can learn from staring at this equation, because the consumption schedule is endogenous and depends on the tax schedule. So we have taxes on both sides of the equation.

What if we assume $u(c) = c$? (very non-standard in macro, very common in public finance)

Now our tax expression simplifies to

$$\frac{\tau(\theta)}{1-\tau(\theta)} = (1+\sigma)\frac{1}{\theta f(\theta)}\int_{s=\theta}^{\infty}\left(1-\frac{W(s)}{\int W(i)f(i)di}\right)dF(\theta)$$

If $W(s)=1$ for all s, then zero marginal tax rates are optimal.

If $W(s)$ is decreasing in s, then want positive tax rates.

Suppose $W(s) = \frac{s^{-\varphi}}{\int s^{-\varphi}f(i)di}$

$$\frac{\tau(\theta)}{1-\tau(\theta)} = (1+\sigma)\frac{[1-F(\theta)]}{\theta f(\theta)}\left(1-\frac{1}{[1-F(\theta)]}\int_{s=\theta}^{\infty}\frac{s^{-\varphi}}{\int s^{-\varphi}f(i)di}dF(\theta)\right)$$

6 Homework

Economy with three types

Productivity	0.5	1.0	2.5
Population share	0.3	0.6	0.1

Utility function is

$$\log c - \frac{h^{1+\sigma}}{1+\sigma}$$

$\sigma = 2$

No government purchases

Solve for optimal allocations for (i) utilitarian planner, (ii) planner that only cares about one of the 3 types, for each of the three types