Notes on Straub and Werning (very rough)

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Judd 1985 economy

Two differences relative to rep agent we discussed last time

Two types of agents: capitalists, who don't work + workers who can't save

Initial capital stock k_0

Inelastic labor supply

No government bonds => period by period budget constraint

Constant spending g

Planner's instruments: capital taxes, and lump-sum transfers to / from workers

Planner's obective

$$\max \Sigma \beta^t \{u(c_t) + \gamma U(C_t)\}$$

with a focus on special case $\gamma = 0$

Workers

$$c_t = w_t + T_t$$

Capitalists

$$\max \Sigma \beta^t U(C_t)$$

$$C_t + a_{t+1} = R_t a_t$$

FOC

$$U'(C_{t-1}) = \beta R_t U'(C_t)$$

where

$$\begin{aligned} R_t &= (1 - tax_t) R_t^* \\ R_t^* &= 1 + r_t \end{aligned}$$

Firms

$$\max f(k_t, l_t) - r_t k_t - w_t l_t$$

FOCs

$$\begin{aligned} f_k(k_t) &= r_t \\ f_l(l_t) &= w_t \end{aligned}$$

Market clearing

$$\begin{aligned} l_t &= 1 \\ a_t &= k_t \end{aligned}$$

which implies

$$w_t = (1-\theta)f(k_t) = f(k_t) - f'(k_t)k$$

GBC

$$g + T_t = (R_t^* - R_t)k_t$$

where

$$R_t^* = (1 + r_t)$$

so

$$(R_t^* - R_t)k_t = tax_t(1 + r_t)k_t.$$

Use BC + mkt clearing to substitute out for R_{t+1} in FOC => Implementability constraint

$$U'(C_{t-1})k_t = \beta(C_t + k_{t+1})U'(C_t)$$

This plus resource constraint

$$c_t + C_t + g + k_{t+1} = f(k_t) + (1-\delta)k_t$$

summarize all the equilibrium conditions.

Why is this so? IC embeds capitalist's budget constraint and FOC. If planner's choices are consistent with those that capitalist wants to choose, and are resource feasible, then they can be implemented, because workers make no choices.

Planner's problem

$$\max \sum_t \beta^t \{u(c_t) + \gamma U(C_t)\}$$

$$\begin{aligned} \beta^t \lambda_t &: \quad c_t + C_t + g + k_{t+1} \leq f(k_t) + (1-\delta)k_t \\ \beta^t \mu_t &: \quad \beta U'(C_t)(C_t + k_{t+1}) = U'(C_{t-1})k_t \end{aligned}$$

$$\begin{aligned} u'(c_t) - \lambda_t &= 0 \\ \gamma U'(C_t) - \lambda_t + \mu_t \beta U''(C_t)(C_t + k_{t+1}) + \mu_t \beta U'(C_t) - \mu_{t+1}\beta U''(C_t)k_{t+1} &= 0 \\ -\lambda_t + \beta\lambda_{t+1}(f'(k_t) + (1-\delta)) + \mu_t \beta U'(C_t) - \beta\mu_{t+1}U'(C_t) &= 0 \end{aligned}$$

Now note from the first and third FOCs that if the optimal policy converges to an interior solution with constant and finite values for allocations and multipliers, then it immediate that in that steady state $tax_t = 0$. To get that result we can ignore the second FOC altogether, and thus also ignore the value for γ.

But is that the right solution?
Log utility case
Here things simplify
Guess (and verify)

$$k_{t+1} = \alpha R_t k_t$$

So

$$\begin{aligned} C_t &= R_t k_t - k_{t+1} \\ &= (1-\alpha) R_t k_t \end{aligned}$$

So

$$\begin{aligned} \frac{C_{t+1}}{C_t} &= \frac{(1-\alpha) R_{t+1} \alpha R_t k_t}{(1-\alpha) R_t k_t} = R_{t+1}\alpha = \beta R_{t+1} \\ &= > \alpha = \beta \end{aligned}$$

So now we can combine the budget constraint and the FOC for the capitalist to get the implementality condition

$$\begin{aligned} C_t + k_{t+1} &= R k_t \\ k_{t+1} &= \beta R_t k_t \end{aligned}$$

$$k_{t+1} = \frac{\beta}{1-\beta} C_t$$

which gives us

$$C_t = \frac{1-\beta}{\beta} k_{t+1}$$

as an implementability constraint, which we can just substitute directly into the problem. Then, with zero concern for capitalists, the Ramsey problem becomes:

$$\begin{aligned} & \max \sum \beta^t u(c_t) \\ c_t + \frac{1-\beta}{\beta} k_{t+1} + k_{t+1} &= f(k_t) + (1-\delta) k_t \end{aligned}$$

FOC wrt k_{t+1} is

$$\begin{aligned} -\mu_t \frac{1}{\beta} &= \mu_{t+1} \left(1 - \delta + f'(k_{t+1})\right) \\ \mu_t &= \beta^t u'(c_t) \\ \mu_{t+1} &= \beta^{t+1} u'(c_{t+1}) \\ u'(c_t) &= \beta^2 u'(c_{t+1}) \left(1 - \delta + f'(k_{t+1})\right) \end{aligned}$$

Compare this to

$$u'(c_t) = \beta u'(c_{t+1}) R_{t+1}$$

where

$$R_{t+1} = (1 - tax_{t+1})\,(1 + f'(k_{t+1}) - \delta)$$

Clearly $tax_{t+1} = 1 - \beta$.

So something must have gone wrong in the general case when we imposed that everything converged.

Suppose

$$\begin{aligned}
U(C) &= \frac{C^{1-\sigma}}{1-\sigma} \\
U'(C) &= C^{-\sigma} \\
U''(C) &= -\sigma C^{-\sigma-1} \\
U''(C)C &= -\sigma C^{-\sigma} = -\sigma U'(C)
\end{aligned}$$

Let

$$\begin{aligned}
\kappa_t &= \frac{k_t}{C_{t-1}} \\
\upsilon_t &= \frac{U'(C_t)}{u'(c_t)}
\end{aligned}$$

So the second condition can be written as

$$\begin{aligned}
\gamma U'(C_t) - \lambda_t + \mu_t \beta U''(C_t)(C_t + k_{t+1}) + \mu_t \beta U'(C_t) - \mu_{t+1} U''(C_t) k_{t+1} &= 0 \\
\gamma \frac{U'(C_t)}{u'(c_t)} - 1 - \sigma \mu_t \beta \frac{U'(C_t)}{u'(c_t)} - \sigma \mu_t \beta \frac{U'(C_t)}{u'(c_t)} \frac{k_{t+1}}{C_t} + \mu_t \beta \frac{U'(C_t)}{u'(c_t)} + \sigma \mu_{t+1} \frac{U'(C_t)}{u'(c_t)} \frac{k_{t+1}}{C_t} &= 0 \\
\gamma \upsilon_t - 1 - \sigma \mu_t \beta \upsilon_t - \sigma \mu_t \beta \upsilon_t \kappa_{t+1} + \mu_t \beta \upsilon_t + \sigma \mu_{t+1} \upsilon_t \kappa_{t+1} &= 0
\end{aligned}$$

$$\begin{aligned}
\sigma \mu_{t+1} \upsilon_t \kappa_{t+1} &= -\gamma \upsilon_t + 1 + \sigma \mu_t \beta \upsilon_t + \sigma \mu_t \beta \upsilon_t \kappa_{t+1} - \mu_t \beta \upsilon_t \\
\beta \mu_{t+1} &= \frac{-\gamma \upsilon_t + 1 + \sigma \mu_t \beta \upsilon_t + \sigma \mu_t \beta \upsilon_t \kappa_{t+1} - \mu_t \beta \upsilon_t}{\sigma \upsilon_t \kappa_{t+1}} \\
\mu_{t+1} &= \mu_t \left(\frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{(1 - \gamma \upsilon_t)}{\beta \sigma \upsilon_t \kappa_{t+1}}
\end{aligned}$$

Now take the other FOC

$$\begin{aligned}
-u'(c_t) + \beta u'(c_{t+1})(f'(k_t) + (1-\delta)) + \mu_t U'(C_t) - \beta \mu_{t+1} U'(C_t) &= 0 \\
-1 + \beta \frac{u'(c_{t+1})}{u'(c_t)} (f'(k_t) + (1-\delta)) + \mu_t \upsilon_t - \beta \mu_{t+1} \upsilon_t &= 0 \\
\frac{u'(c_{t+1})}{u'(c_t)} (f'(k_t) + (1-\delta)) &= \frac{\beta \mu_{t+1} \upsilon_t - \beta \mu_t \upsilon_t + 1}{\beta} \\
&= \frac{1}{\beta} + \upsilon_t \left(\mu_{t+1} - \mu_t \right)
\end{aligned}$$

Go back to second FOC and impose $\gamma = 0$

$$\mu_{t+1} = \mu_t \left(\frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{1}{\beta \sigma \upsilon_t \kappa_{t+1}}$$

It is immediate that when $\sigma > 1$ that μ_t grows without bound – and so does $(\mu_{t+1} - \mu_t)$

Then the last term in the capital FOC is not zero.

Let's suppose k_t converges to something. (cannot go to zero with $g > 0$ or explode)

Then C_t converges too.

But then c_t cannot converge to a finite value, because that would mean μ_t converged, and we would be back at the zero capital tax result – which we know is wrong

So c_t must converge to zero

Also we know that

$$\begin{aligned} C + k &= Rk \\ 1 &= \beta R \end{aligned}$$

so

$$\begin{aligned} C &= (R-1)k \\ &= \frac{1-\beta}{\beta}k \end{aligned}$$

So k_t converges to solution to

$$\frac{1}{\beta}k_g + g = f(k_g) + (1-\delta)k_g$$

with C_t converging to

$$\begin{aligned} C_g &= f(k_g) - g - \delta k_g \\ &= \frac{1-\beta}{\beta}k_g \end{aligned}$$

What capital tax rate supports this? We have

$$1 = \beta(1 - tax)(1 + f'(k_g) - \delta)$$

We can solve the capital steady state equation for k_g given g and then we immediately have the steady state tax rate from this last equation.