Life-Cycle Taxation with Persistent Shocks

A Recursive Formulation*

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1 Introduction

The mechanism design approach to dynamic taxation is also used to study optimal taxation of earnings and capital over the life-cycle. In this setup, the stochastic process for labor productivity is assumed persistent. Persistent shocks to labor productivity are used to capture the observed persistence of wage rates and labor earnings as emphasized by the empirical labor literature. The planner problems in this literature (e.g. Farhi and Werning (2013), Golosov, Troshkin and Tsyvinski (2016), Stantcheva (2018), N'Diaye (2018)) are formulated and solved recursively.

In this note we extend the example on Golosov, Kocherlakota and Tsyvinski (2003) and the inverse Euler equation to derive the recursive formulation of the insurer's maximization problem using techniques due to Fernandes and Phelan (2000). Specifically, we use a two period example to show the use of promise-keeping conditions and threat-keeping conditions to write the problem recursively.

2 Life-Cycle Problem

Skill Process. The model features two periods, period 0 and 1. Labor productivity in the initial period is denoted $\theta_0 \in \{\theta_L, \theta_H\}$, with $\theta_L < \theta_H$. We use π_j to denote the probability of being type $j \in \{H, L\}$ in the initial period. Labor productivity evolves between the initial and final period, with final period productivity $\theta_1 \in \{\theta_L, \theta_H\}$. Let $\pi_{i|j}$ describe the conditional probability of realizing type i in the final

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period after being type j in the initial period. The ordering of the subscripts on the allocations is chronological: the first index refers to the initial period, the second script to the final period.

Preferences. To illustrate another useful technique for this literature, I assume preferences are separable in consumption c and hours worked $\ell = y/\theta$ and given by:

$$u\big(c, y; \theta\big) = u(c) - \ell^{1+\frac{1}{\eta}} \ ,$$

where u is increasing and strictly concave, and where $\eta > 0$ is the Frisch elasticity of labor supply.

Insurer Problem. The profit-maximizing insurer has access to a linear savings technology with gross return rate R and chooses $(c_H, y_H, c_L, y_L, c_{HH}, y_{HH}, c_{HL}, y_{HL}, c_{LH}, y_{LH}, c_{LL}, y_{LL})$ to solve:

$$\begin{aligned}\max \ &\pi_H\left(y_H - c_H\right) + \frac{1}{R}\Big[\pi_H\big(\pi_{H|H}\left(y_{HH} - c_{HH}\right) + \pi_{L|H}\left(y_{HL} - c_{HL}\right)\big)\Big] \\ &+\pi_L\left(y_L - c_L\right) + \frac{1}{R}\Big[\pi_L\big(\pi_{H|L}\left(y_{LH} - c_{LH}\right) + \pi_{L|L}\left(y_{LL} - c_{LL}\right)\big)\Big]\end{aligned}$$

subject to an ex-ante welfare constraint:

$$\begin{aligned}&\pi_H u(c_H, y_H/\theta_H) + \beta\pi_H\big[\pi_{H|H}u(c_{HH}, y_{HH}/\theta_H) + \pi_{L|H}u(c_{HL}, y_{HL}/\theta_L)\big] \\ &+\pi_L u(c_L, y_L/\theta_L) + \beta\pi_L\big[\pi_{H|L}u(c_{LH}, y_{LH}/\theta_H) + \pi_{L|L}u(c_{LL}, y_{LL}/\theta_L)\big] \geq \mathcal{V} \ ,\end{aligned}$$

where $\mathcal{V}$ is the ex-ante welfare promise. The insurer's maximization problem is constrained by four incentive constraints. First, the insurer dissuades high productivity types from misreporting in the initial period (assuming truthful reporting in the final period):

$$\begin{aligned}&u(c_H, y_H/\theta_H) + \beta\Big[\pi_{H|H}u(c_{HH}, y_{HH}/\theta_H) + \pi_{L|H}u(c_{HL}, y_{HL}/\theta_L)\Big] \\ &\geq u(c_L, y_L/\theta_H) + \beta\Big[\pi_{H|H}u(c_{LH}, y_{LH}/\theta_H) + \pi_{L|H}u(c_{LL}, y_{LL}/\theta_L)\Big] \ .\end{aligned}$$

where it is should be noted that final period utilities are evaluated at the true probabilities. Similarly, the low productivity type is discouraged to misreport in the initial period (in practice we might not need to worry about this constraint, but we will include it for now):

$$\begin{aligned}&u(c_L, y_L/\theta_L) + \beta\Big[\pi_{H|L}u(c_{LH}, y_{LH}/\theta_H) + \pi_{L|L}u(c_{LL}, y_{LL}/\theta_L)\Big] \\ &\geq u(c_H, y_H/\theta_L) + \beta\Big[\pi_{H|L}u(c_{HH}, y_{HH}/\theta_H) + \pi_{L|L}u(c_{HL}, y_{HL}/\theta_L)\Big] \ .\end{aligned}$$

Furthermore, the solution to the insurer problem is restricted to respect the incentive constraints for high types in the final period, given their report in the initial period (we won't worry about low types misreporting in the second period).

$$u(c_{HH}, y_{HH}/\theta_H) \geq u(c_{HL}, y_{HL}/\theta_H) \; ;$$
$$u(c_{LH}, y_{LH}/\theta_H) \geq u(c_{LL}, y_{LL}/\theta_H) \; .$$

Note the following about the formulation of the insurer problem:

1. We assume that the IC constraints for the high productivity types are binding in the final period. *Question*: Is this without loss of generality? If yes, prove it. If not, construct a counterexample.

2. We assume that after misreporting in the initial period, the agent reports truthfully in the final period (in the initial period IC constraints). We could consider misreporting in both periods. *Question*: Is our formulation without loss? If yes, prove it. If not, construct a counterexample.

Reformulated Insurer Problem. Before writing out the optimality conditions to the insurer problem, we examine whether the insurer problem has a unique solution. To verify this, reformulate the problem in utility space.[1] To do so, define $\bar{u}_i \equiv u(c_i)$ and $\bar{h}_i \equiv \left(\frac{y_i}{\theta_j}\right)^{1+\frac{1}{\eta}}$, where $i \in \{H, L, HH, HL, LH, LL\}$, and $j = L$ for $i \in \{L, HL, LL\}$ and $j = H$ otherwise. These definitions imply $c_i = u^{-1}(\bar{u}_i)$ and $y_i = \theta_j \bar{h}_i^{\frac{1}{1+\frac{1}{\eta}}}$.

The insurer chooses $(\bar{u}_H, \bar{h}_H, \bar{u}_L, \bar{h}_L, \bar{u}_{HH}, \bar{h}_{HH}, \bar{u}_{HL}, \bar{h}_{HL}, \bar{u}_{LH}, \bar{h}_{LH}, \bar{u}_{LL}, \bar{h}_{LL})$ to solve:

$$\max\ \pi_H\left(\theta_H \bar{h}_H^{\frac{1}{1+\frac{1}{\eta}}} - u^{-1}(\bar{u}_H)\right) + \frac{1}{R}\left[\pi_H\left(\pi_{H|H}\left(\theta_H \bar{h}_{HH}^{\frac{1}{1+\frac{1}{\eta}}} - u^{-1}(\bar{u}_{HH})\right) + \pi_{L|H}\left(\theta_L \bar{h}_{HL}^{\frac{1}{1+\frac{1}{\eta}}} - u^{-1}(\bar{u}_{HL})\right)\right)\right]$$
$$+\pi_L\left(\theta_L \bar{h}_L^{\frac{1}{1+\frac{1}{\eta}}} - u^{-1}(\bar{u}_L)\right) + \frac{1}{R}\left[\pi_L\left(\pi_{H|L}\left(\theta_H \bar{h}_{LH}^{\frac{1}{1+\frac{1}{\eta}}} - u^{-1}(\bar{u}_{LH})\right) + \pi_{L|L}\left(\theta_L \bar{h}_{LL}^{\frac{1}{1+\frac{1}{\eta}}} - u^{-1}(\bar{u}_{LL})\right)\right)\right]$$

subject to:

$$\pi_H\left(\bar{u}_H - \bar{h}_H\right) + \beta\pi_H\left[\pi_{H|H}\left(\bar{u}_{HH} - \bar{h}_{HH}\right) + \pi_{L|H}\left(\bar{u}_{HL} - \bar{h}_{HL}\right)\right]$$
$$+\pi_L\left(\bar{u}_L - \bar{h}_L\right) + \beta\pi_L\left[\pi_{H|L}\left(\bar{u}_{LH} - \bar{h}_{LH}\right) + \pi_{L|L}\left(\bar{u}_{LL} - \bar{h}_{LL}\right)\right] \geq \mathcal{V}\ ,$$

and four incentive compatibility constraints:

$$\bar{u}_H - \bar{h}_H + \beta\Big[\pi_{H|H}\left(\bar{u}_{HH} - \bar{h}_{HH}\right) + \pi_{L|H}\left(\bar{u}_{HL} - \bar{h}_{HL}\right)\Big]$$
$$\geq \bar{u}_L - \left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\eta}} \bar{h}_L + \beta\Big[\pi_{H|H}\left(\bar{u}_{LH} - \bar{h}_{LH}\right) + \pi_{L|H}\left(\bar{u}_{LL} - \bar{h}_{LL}\right)\Big]\ ,$$

the incentive constraint for the low type in the initial period:

$$\bar{u}_L - \bar{h}_L + \beta\Big[\pi_{H|L}\left(\bar{u}_{LH} - \bar{h}_{LH}\right) + \pi_{L|L}\left(\bar{u}_{LL} - \bar{h}_{LL}\right)\Big]$$
$$\geq \bar{u}_H - \left(\frac{\theta_H}{\theta_L}\right)^{1+\frac{1}{\eta}} \bar{h}_H + \beta\Big[\pi_{H|L}\left(\bar{u}_{HH} - \bar{h}_{HH}\right) + \pi_{L|L}\left(\bar{u}_{HL} - \bar{h}_{HL}\right)\Big]\ ,$$

and the incentive constraints for high types in the final period:

$$\bar{u}_{HH} - \bar{h}_{HH} \geq \bar{u}_{HL} - \left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\eta}} \bar{h}_{HL}\ ;$$
$$\bar{u}_{LH} - \bar{h}_{LH} \geq \bar{u}_{LL} - \left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\eta}} \bar{h}_{LL}\ .$$

The insurer problem has a strictly concave objective and a linear constraint set. The objective function is strictly concave because the sum of two strictly concave functions is strictly concave: $\bar{h}_i^{\frac{1}{1+\frac{1}{\eta}}}$ is strictly concave for $\eta > 0$ and u^{-1} is strictly convex given that u is increasing and strictly concave. If the insurer problem has a unique solution, it is unique.

[1] Also see Rustichini and Phelan (2017).

Return to our original problem: The profit-maximizing insurer chooses $(c_H, y_H, c_L, y_L, c_{HH}, y_{HH}, c_{HL}, y_{HL}, c_{LH}$ to solve:

$$\max\ \pi_H\left(y_H - c_H\right) + \frac{1}{R}\Big[\pi_H\big(\pi_{H|H}\left(y_{HH} - c_{HH}\right) + \pi_{L|H}\left(y_{HL} - c_{HL}\right)\big)\Big]$$
$$+\pi_L\left(y_L - c_L\right) + \frac{1}{R}\Big[\pi_L\big(\pi_{H|L}\left(y_{LH} - c_{LH}\right) + \pi_{L|L}\left(y_{LL} - c_{LL}\right)\big)\Big]$$

subject to (with multplier labels attached to each constraint)

$$\eta : \pi_H u(c_H, y_H/\theta_H) + \beta\pi_H\left[\pi_{H|H}u(c_{HH}, y_{HH}/\theta_H) + \pi_{L|H}u(c_{HL}, y_{HL}/\theta_L)\right]$$
$$+\pi_L u(c_L, y_L/\theta_L) + \beta\pi_L\left[\pi_{H|L}u(c_{LH}, y_{LH}/\theta_H) + \pi_{L|L}u(c_{LL}, y_{LL}/\theta_L)\right] \geq \mathcal{V}\ ,$$

$$\mu_H : u(c_H, y_H/\theta_H) + \beta\Big[\pi_{H|H}u(c_{HH}, y_{HH}/\theta_H) + \pi_{L|H}u(c_{HL}, y_{HL}/\theta_L)\Big]$$
$$\geq u(c_L, y_L/\theta_H) + \beta\Big[\pi_{H|H}u(c_{LH}, y_{LH}/\theta_H) + \pi_{L|H}u(c_{LL}, y_{LL}/\theta_L)\Big]\ .$$

$$\mu_L : u(c_L, y_L/\theta_L) + \beta\Big[\pi_{H|L}u(c_{LH}, y_{LH}/\theta_H) + \pi_{L|L}u(c_{LL}, y_{LL}/\theta_L)\Big]$$
$$\geq u(c_H, y_H/\theta_L) + \beta\Big[\pi_{H|L}u(c_{HH}, y_{HH}/\theta_H) + \pi_{L|L}u(c_{HL}, y_{HL}/\theta_L)\Big]\ .$$

$$\beta\pi_H\mu_{HH} : u(c_{HH}, y_{HH}/\theta_H) \geq u(c_{HL}, y_{HL}/\theta_H)\ ;$$
$$\beta\pi_L\mu_{LH} : u(c_{LH}, y_{LH}/\theta_H) \geq u(c_{LL}, y_{LL}/\theta_H)\ .$$

First-order Conditions. The optimality conditions with respect to (c_H, c_L, y_H, y_L) are respectively given by:

$$1 = u'(c_H)\left(\eta + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\right)$$
$$1 = u'(c_L)\left(\eta - \frac{\mu_H}{\pi_L} + \frac{\mu_L}{\pi_L}\right)$$
$$1 = \left(1 + \frac{1}{\eta}\right)\left(\frac{y_H}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}\left(\eta + \frac{\mu_H}{\pi_H}\right) - \frac{\mu_L}{\pi_H}\left(1 + \frac{1}{\eta}\right)\left(\frac{y_H}{\theta_L}\right)^{\frac{1}{\eta}}\frac{1}{\theta_L}$$
$$1 = \left(1 + \frac{1}{\eta}\right)\left(\frac{y_L}{\theta_L}\right)^{\frac{1}{\eta}}\frac{1}{\theta_L}\left(\eta + \frac{\mu_L}{\pi_L}\right) - \frac{\mu_H}{\pi_L}\left(1 + \frac{1}{\eta}\right)\left(\frac{y_L}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}$$

For consumption in the final period, $(c_{HH}, c_{HL}, c_{LH}, c_{LL})$, the corresponding optimality conditions are:

$$\frac{1}{\beta R} = u'(c_{HH})\left(\eta + \frac{\mu_{HH}}{\pi_{H|H}} + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\frac{\pi_{H|L}}{\pi_{H|H}}\right)$$
$$\frac{1}{\beta R} = u'(c_{HL})\left(\eta - \frac{\mu_{HH}}{\pi_{L|H}} + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\frac{\pi_{L|L}}{\pi_{L|H}}\right)$$
$$\frac{1}{\beta R} = u'(c_{LH})\left(\eta + \frac{\mu_{LH}}{\pi_{H|L}} - \frac{\mu_H}{\pi_L}\frac{\pi_{H|H}}{\pi_{H|L}} + \frac{\mu_L}{\pi_L}\right)$$
$$\frac{1}{\beta R} = u'(c_{LL})\left(\eta - \frac{\mu_{LH}}{\pi_{L|L}} - \frac{\mu_H}{\pi_L}\frac{\pi_{L|H}}{\pi_{L|L}} + \frac{\mu_L}{\pi_L}\right) .$$

The optimality conditions with respect to consumption can be used to establish that $c_{HH} \geq c_{HL}$ and that $c_{LH} \geq c_{LL}$. Given that the incentive constraints for high types in the second period hold with equality, this implies $y_{HH} \geq y_{HL}$ as well as $y_{LH} \geq y_{LL}$.

The optimality conditions with respect to labor in the final period $(y_{HH}, y_{HL}, y_{LH}, y_{LL})$ are:

$$\frac{1}{\beta R} = \left(1+\frac{1}{\eta}\right)\left(\frac{y_{HH}}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}\left(\eta + \frac{\mu_{HH}}{\pi_{H|H}} + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\frac{\pi_{H|L}}{\pi_{H|H}}\right)$$
$$\frac{1}{\beta R} = \left(1+\frac{1}{\eta}\right)\left(\frac{y_{HL}}{\theta_L}\right)^{\frac{1}{\eta}}\frac{1}{\theta_L}\left(\eta + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\frac{\pi_{L|L}}{\pi_{L|H}}\right) - \left(1+\frac{1}{\eta}\right)\left(\frac{y_{HL}}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}\frac{\mu_{HH}}{\pi_{L|H}}$$
$$\frac{1}{\beta R} = \left(1+\frac{1}{\eta}\right)\left(\frac{y_{LH}}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}\left(\eta + \frac{\mu_{LH}}{\pi_{H|L}} - \frac{\mu_H}{\pi_L}\frac{\pi_{H|H}}{\pi_{H|L}} + \frac{\mu_L}{\pi_L}\right)$$
$$\frac{1}{\beta R} = \left(1+\frac{1}{\eta}\right)\left(\frac{y_{LL}}{\theta_L}\right)^{\frac{1}{\eta}}\frac{1}{\theta_L}\left(\eta + \frac{\mu_L}{\pi_L} - \frac{\mu_H}{\pi_L}\frac{\pi_{L|H}}{\pi_{L|L}}\right) - \left(1+\frac{1}{\eta}\right)\left(\frac{y_{LL}}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}\frac{\mu_{LH}}{\pi_{L|L}} .$$

By using the optimality conditions with respect to consumption and labor, we establish that the marginal decision for the high types in the second period is undistorted.

3. *Practice Question*: Derive the inverse Euler equation/equations for this setup.

   This was Job's question. We will answer it now. Take these three FOCs for c_H, c_{HH} and c_{HL} :

$$1 = u'(c_H)\left(\eta + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\right)$$

$$\frac{1}{\beta R} = u'(c_{HH})\left(\eta + \frac{\mu_{HH}}{\pi_{H|H}} + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\frac{\pi_{H|L}}{\pi_{H|H}}\right)$$
$$\frac{1}{\beta R} = u'(c_{HL})\left(\eta - \frac{\mu_{HH}}{\pi_{L|H}} + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\frac{\pi_{L|L}}{\pi_{L|H}}\right)$$

$$
\begin{aligned}
\frac{1}{u'(c_H)} &= \left(\eta + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\right) \\
\frac{1}{\beta R}\left(\frac{\pi_{H|H}}{u'(c_{HH})} + \frac{\pi_{L|H}}{u'(c_{HL})}\right) &= \pi_{H|H}\left(\eta + \frac{\mu_{HH}}{\pi_{H|H}} + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\frac{\pi_{H|L}}{\pi_{H|H}}\right) + \pi_{L|H}\left(\eta - \frac{\mu_{HH}}{\pi_{L|H}} + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\frac{\pi_{L|L}}{\pi_{L|H}}\right) \\
&= \pi_{H|H}\eta + \mu_{HH} + \frac{\pi_{H|H}\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\pi_{H|L} + \pi_{L|H}\eta - \mu_{HH} + \frac{\pi_{L|H}\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\pi_{L|L} \\
&= \left(\eta + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\right)
\end{aligned}
$$

So

$$
\frac{1}{u'(c_H)} = \beta R\left(\frac{\pi_{H|H}}{u'(c_{HH})} + \frac{\pi_{L|H}}{u'(c_{HL})}\right)
$$

This is the famous inverse Euler equation. Suppose utility is logarithmic. Then we have

$$
c_H = \beta R\left(\pi_{H|H}c_{HH} + \pi_{L|H}c_{HL}\right)
$$

Suppose we want to define the implicit savings wedge for this allocation as the tax rate that would satisfy a standard optimal savings condition. Thus the wedge is the solution $1-\tau$ to

$$
\begin{aligned}
u'(c_H) &= \beta R(1-\tau)\left(\pi_{H|H}u'(c_{HH}) + \pi_{L|H}u'(c_{HL})\right) \\
\frac{1}{c_H} &= \beta R(1-\tau)\left(\pi_{H|H}\frac{1}{c_{HH}} + \pi_{L|H}\frac{1}{c_{HL}}\right)
\end{aligned}
$$

Suppose $\beta R = 1$. If this decentralization decentralizes the optimal allocation, then combining the inverse Euler and the decentralization FOC we have

$$
\frac{1}{E[c']} = (1-\tau)E\left[\frac{1}{c'}\right]
$$

Jensen's inequality states that if f is convex, $E[f(x)] > f(E[x])$. Now $\frac{1}{x}$ is a convex function, so $E\left[\frac{1}{c'}\right] > \frac{1}{E[c']}$, which implies $\tau > 0$. Thus, the optimum implicitly taxes saving.

Recursive Problem: Profit Maximization. This two period example was fine, except that even with only two periods we ended up with many first-order conditions. Imagine trying to compute the optimal consumption and labor allocations for any possible history with multiple periods and multiple values for θ. The problem will explode unless we can formulate it recursively.

So let's write this problem in a recursive fashion, using the techniques of Fernandes and Phelan (2000). To start, let's stick with the assumption that there are only two periods. We solve the planner problem after invoking the standard argument that the incentive constraints for the second period high types has to bind.

In the initial period, the planner chooses allocation (c_H, y_H, c_L, y_L), promised utilities (V_H, V_L), and threat utilities (V_{HL}, V_{LH}). Threat utility V_{HL} is the promised utility to innate type H that reports L. We need this object because we need to make sure that misreporting does not pay off in terms of lifetime utility. And just as the planner can manipulate promised future values to make truthtelling more attractive, so can she manipulate promised future values conditional on lying today in order to make lying less attractive.

The insurer thus solves:

$$\max \ \ \pi_H \left(y_H - c_H\right) + \pi_L \left(y_L - c_L\right) + \frac{1}{R}\Big[\pi_H \Pi \left(V_H, V_{LH}, \theta_H\right) + \pi_L \Pi \left(V_L, V_{HL}, \theta_L\right)\Big] \ ,$$

where Π is the value function for the final period which we discuss below, subject to the ex-ante welfare constraint:

$$\pi_H U(c_H, y_H/\theta_H) + \pi_L U(c_L, y_L/\theta_L) + \beta\big[\pi_H V_H + \pi_L V_L\big] \geq \mathcal{V} \ ,$$

where $\mathcal{V}$ is the endowment of ex-ante welfare. Furthermore, maximization of profits is initially constrained by the two incentive compatibility constraints for the initial period:

$$\begin{aligned}
0 &\leq U(c_H, y_H/\theta_H) - U(c_L, y_L/\theta_H) + \beta\big[V_H - V_{HL}\big] \\
0 &\leq U(c_L, y_L/\theta_L) - U(c_H, y_H/\theta_L) + \beta\big[V_L - V_{LH}\big] \ .
\end{aligned}$$

Comparing these expressions to the initial period incentive constraints gives an interpretation for the promised utilities and threat utilities. The promised utility V_H is the expected utility delivered in the final period when the individual reports truthfully today. When the high productivity type reports L instead, the expected utility is V_{HL}. The planner chooses the threat value to prevent misreporting in the initial period.

The constraints are respectively given multipliers η, μ_H, μ_L. As a result, the optimality conditions to the initial period allocation (c_H, y_H, c_L, y_L) are given by:

$$\begin{aligned}
1 &= u'(c_H)\left(\eta + \frac{\mu_H}{\pi_H} - \frac{\mu_L}{\pi_H}\right) \\
1 &= u'(c_L)\left(\eta - \frac{\mu_H}{\pi_L} + \frac{\mu_L}{\pi_L}\right) \\
1 &= \left(1+\frac{1}{\eta}\right)\left(\frac{y_H}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}\left(\eta + \frac{\mu_H}{\pi_H}\right) - \frac{\mu_L}{\pi_H}\left(1+\frac{1}{\eta}\right)\left(\frac{y_H}{\theta_L}\right)^{\frac{1}{\eta}}\frac{1}{\theta_L} \\
1 &= \left(1+\frac{1}{\eta}\right)\left(\frac{y_L}{\theta_L}\right)^{\frac{1}{\eta}}\frac{1}{\theta_L}\left(\eta + \frac{\mu_L}{\pi_L}\right) - \frac{\mu_H}{\pi_L}\left(1+\frac{1}{\eta}\right)\left(\frac{y_L}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H} \ ,
\end{aligned}$$

(which are exactly the same conditions we had for the previous formulation) while the optimality conditions with respect to future values $(V_H, V_L, V_{HL}, V_{LH})$ are:

$$\beta R \left(\eta + \frac{\mu_H}{\pi_H} \right) = -\Pi_1 \left(V_H, V_{LH}, \theta_H \right)$$
$$\beta R \left(\eta + \frac{\mu_L}{\pi_L} \right) = -\Pi_1 \left(V_L, V_{HL}, \theta_L \right)$$
$$\beta R \frac{\mu_H}{\pi_L} = \Pi_2 \left(V_L, V_{HL}, \theta_L \right)$$
$$\beta R \frac{\mu_L}{\pi_H} = \Pi_2 \left(V_H, V_{LH}, \theta_H \right)$$

For the final period, we solve two separate component problems given the state variables promised utility, threat utility, and the past realization of labor productivity type. When the initial productivity level is θ_H, the final period planner problem chooses $(c_{HH}, y_{HH}, c_{HL}, y_{HL})$ to solve:

$$\Pi \left(V_H, V_{LH}, \theta_H \right) \equiv \max \ \pi_{H|H} \left(y_{HH} - c_{HH} \right) + \pi_{L|H} \left(y_{HL} - c_{HL} \right) \ ,$$

subject to promise keeping, threat keeping, and the incentive compatibility condition:

$$\pi_{H|H} U(c_{HH}, y_{HH}/\theta_H) + \pi_{L|H} U(c_{HL}, y_{HL}/\theta_L) \geq V_H$$
$$\pi_{H|L} U(c_{HH}, y_{HH}/\theta_H) + \pi_{L|L} U(c_{HL}, y_{HL}/\theta_L) \leq V_{LH}$$
$$U(c_{HH}, y_{HH}/\theta_H) \geq U(c_{HL}, y_{HL}/\theta_H)$$

The multipliers on the constraints are given by ϕ_H, ϕ_{LH} and $\tilde{\mu}_{HH} \equiv \beta R \mu_{HH}$. Evaluating the promise keeping and threat keeping condition, we observe the key complication faced when assuming persistent shocks: privately observed histories of productivity shocks influence the way in which agents evaluate continuation contracts. The evaluation of contracts designed for the final period $(c_{HH}, y_{HH}, c_{HL}, y_{HL})$ depends on the individual's type in the initial period. When the shocks are instead time independent, the evaluation of future contracts is identical, and the threat keeping constraint is redundant.

The final period insurer problem is constrained in two ways. The promise keeping condition restricts the planner to deliver an allocation delivering utility V_H to agents reporting truthfully. At the same time, the threat keeping conditions ensures that no gains are made by an individual misreported in the initial period, which is ensured by delivering V_{LH}.

The optimality conditions with respect to $(c_{HH}, y_{HH}, c_{HL}, y_{HL})$ are given by:

$$1 = u'(c_{HH})\left(\phi_H + \frac{\pi_{H|L}}{\pi_{H|H}}\phi_{LH} + \frac{\tilde{\mu}_{HH}}{\pi_{H|H}}\right)$$
$$1 = \left(1+\frac{1}{\eta}\right)\left(\frac{y_{HH}}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}\left(\phi_H + \frac{\pi_{H|L}}{\pi_{H|H}}\phi_{LH} + \frac{\tilde{\mu}_{HH}}{\pi_{H|H}}\right)$$
$$1 = u'(c_{HL})\left(\phi_H + \frac{\pi_{L|L}}{\pi_{L|H}}\phi_{LH} - \frac{\tilde{\mu}_{HH}}{\pi_{L|H}}\right)$$
$$1 = \left(1+\frac{1}{\eta}\right)\left(\frac{y_{HL}}{\theta_L}\right)^{\frac{1}{\eta}}\frac{1}{\theta_L}\left(\phi_H + \frac{\pi_{L|L}}{\pi_{L|H}}\phi_{LH}\right) - \left(1+\frac{1}{\eta}\right)\left(\frac{y_{HL}}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}\frac{\tilde{\mu}_{HH}}{\pi_{L|H}}$$

Combining the first-order conditions for type HH, we note that the optimality conditions imply that the marginal decisions for the high type are undistorted (so the zero tax at the top result survives in the last period). In addition, this problem gives rise to the following envelope conditions:

$$\Pi_1(V_H, V_{LH}, \theta_H) = -\phi_H$$
$$\Pi_2(V_H, V_{LH}, \theta_H) = -\phi_{LH}$$

Similarly, we solve the social planner problem given previously realized productivity level is θ_L. The final period insurer chooses $(c_{LH}, y_{LH}, c_{LL}, y_{LL})$ to solve:

$$\Pi(V_L, V_{HL}, \theta_L) \equiv \max \; \pi_{H|L}(y_{LH} - c_{LH}) + \pi_{L|L}(y_{LL} - c_{LL}) \;,$$

subject to promise keeping, threat keeping, and the incentive compatibility condition:

$$\pi_{H|L}U(c_{LH}, y_{LH}/\theta_H) + \pi_{L|L}U(c_{LL}, y_{LL}/\theta_L) \geq V_L$$
$$\pi_{H|H}U(c_{LH}, y_{LH}/\theta_H) + \pi_{L|H}U(c_{LL}, y_{LL}/\theta_L) \leq V_{HL}$$
$$U(c_{LH}, y_{LH}/\theta_H) \geq U(c_{LL}, y_{LL}/\theta_H)$$

The multipliers on the constraints are respectively given by ϕ_L, ϕ_{HL}, and $\tilde{\mu}_{LH} \equiv \beta R \mu_{LH}$.

The optimality conditions with respect to $(c_{LH}, y_{LH}, c_{LL}, y_{LL})$ are given by:

$$1 = u'(c_{LH})\left(\phi_L + \frac{\pi_{H|H}}{\pi_{H|L}}\phi_{HL} + \frac{\tilde{\mu}_{LH}}{\pi_{H|L}}\right)$$
$$1 = \left(1+\frac{1}{\eta}\right)\left(\frac{y_{LH}}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}\left(\phi_L + \frac{\pi_{H|H}}{\pi_{H|L}}\phi_{HL} + \frac{\tilde{\mu}_{LH}}{\pi_{H|L}}\right)$$
$$1 = u'(c_{LL})\left(\phi_L + \frac{\pi_{L|H}}{\pi_{L|L}}\phi_{HL} - \frac{\tilde{\mu}_{LH}}{\pi_{L|L}}\right)$$
$$1 = \left(1+\frac{1}{\eta}\right)\left(\frac{y_{LL}}{\theta_L}\right)^{\frac{1}{\eta}}\frac{1}{\theta_L}\left(\phi_L + \frac{\pi_{L|H}}{\pi_{L|L}}\phi_{HL}\right) - \left(1+\frac{1}{\eta}\right)\left(\frac{y_{LL}}{\theta_H}\right)^{\frac{1}{\eta}}\frac{1}{\theta_H}\frac{\tilde{\mu}_{LH}}{\pi_{L|L}}$$

Note that the optimality conditions imply that the marginal decisions for the high type are undistorted. In addition, this problem gives rise to the following envelope conditions:

$$\Pi_1 (V_L, V_{HL}, \theta_L) = -\phi_L$$
$$\Pi_2 (V_L, V_{HL}, \theta_L) = -\phi_{HL}$$

We fully characterized the solution to the recursive profit maximization problem. Next, we establish that the solution to the recursive profit maximization problem aligns with the sequential profit maximization problem.

Proposition 1. Equivalence Sequential and Recursive Profit Maximization Problem.
$(c_H, y_H, c_L, y_L, c_{HH}, y_{HH}, c_{HL}, y_{HL}, c_{LH}, y_{LH}, c_{LL}, y_{LL}, \eta, \mu_H, \mu_L, \mu_{HH}, \mu_{LH})$ solves the sequential cost minimization problem given $\mathcal{V}$ if and only if the recursive cost minimization problem given $\mathcal{V}$ is solved by $(c_H, y_H, c_L, y_L, c_{HH}, y_{HH}, c_{HL}, y_{HL}, c_{LH}, y_{LH}, c_{LL}, y_{LL}, \eta, \mu_H, \mu_L, \mu_{HH}, \mu_{LH})$.

Proof. For the initial period, we directly observe that the optimality conditions are identical. For the final period, an identical observation is made after using the optimality conditions and envelope conditions with respect to promised utilities and threat utilities. Specifically, we rewrite the optimality conditions with respect to future utilities $(V_H, V_L, V_{HL}, V_{LH})$ as:

$$\beta R \left(\eta + \frac{\mu_H}{\pi_H} \right) = \phi_H$$
$$\beta R \left(\eta + \frac{\mu_L}{\pi_L} \right) = \phi_L$$
$$\beta R \frac{\mu_H}{\pi_L} = -\phi_{HL}$$
$$\beta R \frac{\mu_L}{\pi_H} = -\phi_{LH}$$

Substituting these expressions into the optimality conditions on the previou page shows the optimality conditions are indeed identical to those from the sequence problem. ■

4. We wrote the planner problem as a cost minimization problem subject to a participation constraint for a risk-neutral insurer.

   Question: Is this the dual problem of the welfare maximization problem subject to a resource constraint?

5. Formulate the welfare maximization problem in recursive form.

   Question: Prove equivalence between the sequential welfare maximization problem and the recursive welfare maximization problem.

3 A Formulation with Multiple Values for the Shock and Multiple Periods

The invididual states are: (i) age a, (ii) last period productivity θ_{-1}, (iii) expected promised value from today onwards V

There is also (iv) a threat value $\tilde{V}$ which applies to someone one type more productive than θ_{-1} yesterday – call this type θ_{-1}^{+}. (We could imagine a threat value for any possible false report in the previous period, but we will assume that the only misreport we need to worry about is from a type pretending to one type less productive than they truly are)

If this type misreported yesterday they will be pooled with the θ_{-1} type today.

Call the individual state vector $s = \left(a, \theta_{-1}, V, \tilde{V}\right)$

Prior to the realization of θ, for each point in the individual state space s, the planner chooses $c(\theta, s)$, $y(\theta, s)$, $V(\theta, s)$, $\tilde{V}(\theta, s)$ in order to solve

$$\Pi(s) = \max_{c(\theta,s),y(\theta,s),V(\theta,s),\tilde{V}(\theta,s)} \sum_{\theta} \left\{ \pi(\theta|\theta_{-1}) \left[y(\theta, s) - c(\theta, s) + \frac{1}{R}\Pi(a+1, \theta, V(\theta, s), \tilde{V}(\theta, s)) \right] \right\}$$

subject to promise-keeping and threatkeeping

$$\begin{aligned} \sum_{\theta} \pi(\theta|\theta_{-1}) \left\{ u(c(\theta, s), y(\theta, s)/\theta) + \beta V(\theta, s) \right\} &\geq V \\ \sum_{\theta} \pi(\theta|\theta_{-1}^{+}) \left\{ u(c(\theta, s), u(y(\theta, s)/\theta) + \beta V(\theta, s) \right\} &\leq \tilde{V} \end{aligned}$$

and to local downward incentive constraints

$$u(c(\theta, s), y(\theta, s)/\theta) + \beta V(\theta, s) \geq u(c(\theta^{-}, s), y(\theta^{-}, s)/\theta) + \beta \tilde{V}(\theta^{-}, s) \qquad \theta = 2, ..., N$$

where θ^{-} is one type less productive than θ.

At each age this problem can be solved, one value for s at at time.

Then we can move back one age.

References